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THE DESIGN

OF

PLATE-GIRDERS

THE DESIGN

OF

PLATE-GIRDERS

WITH APPENDICES
ON
'COLUMNS AND STRUTS,' 'ECONOMIC COLUMNS' AND
'WEB STRESSES IN PLATE GIRDERS'

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PREFACE.

The analysis of the stresses in the web of the plate-girder is a complex problem, and does not admit of such easy solution as in the corresponding case of the members of triangulated girders, in which the stresses can be determined with some measure of certainty. This has led to rough-and-ready rules, based upon experience, for the design of the web, with the result that an excess of material is often provided. The usual procedure adopted is to design the web on the assumption that it takes the shear stress alone, and then to consider the struts or stiffeners, their number and disposition being largely a matter of experience.

For plate-girders of small span and depth this does not involve much error; but for girders of long span and moderate depth, it is necessary that the stiffeners or struts should be considered jointly with the web, if economy is to be attained.

In this work the author has briefly set forth the theory of the distribution of stress in the plate-girder, and then shown the close relation which exists between it and the lattice-girder. The web and struts are then considered together, and a rational method for their design given. Further, it is shown that the girder so designed is more economical of material than if designed on the usual assumptions, and that therefore it is a more correct solution of the problem.

Of late years the practice has been to design plate-girders for small spans, and simple triangulated girders for the larger spans. The author is of the opinion that, if the advantages of the compound and lattice types of girders (shown on figs. 55, 56) were appreciated, a revival of girders of these types would take place, and they would be used for much larger spans than hitherto, and to some extent would displace the more expensive triangulated girder. The analysis of the weight of plate-girders, and its application to parallel girders in the form given, is—so far as the author is aware—new: for rectangular, lattice, and triangulated girders, the inclination of the bracing for maximum economy is shown to be 45°.

The attention of the reader is directed to the following sections:—

The equivalent load diagram (§ 26).
The division of the shearing-force diagram into equal areas (§§ 50, 56).
The equivalent length of booms (§ 52).
The effect of compound stress on the web (§§ 45, 46).
The designs of the compound and lattice-girders (§§ 61, 62).
The weight of plate and parallel girders (§§ 65, 70).

The author's thanks are due to Professor Alexander, for his kindly criticism in the preparation of this work, and for the use of some of the blocks from Alexander and Thomson's well-known Treatise on Applied Mechanics; to the Institution of Civil Engineers of Ireland, for permission to make use of the Papers in the *Proceedings* on "The Design of Plate-girders," and "The Weight of Plate-girders"; to Mr. J. P. Conradi, for the abstract of his Paper on "Standard Types of Superstructure"; to Messrs. Dorman, Long, & Co., and to the Motherwell Iron and Bridge Co., for the particulars of the data and sizes of the trough-floors; to Messrs. J. T. Jackson and C. J. Neely, for their kindness in reading the proofs for Press.

WALTER E. LILLY.

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THE DESIGN OF PLATE-GIRDERS.

THE STRENGTHS OF IRON AND STEEL.

1. **Materials.**—The materials principally used in the construction of plate and rolled girders are mild steel and wrought iron, of which the former is the more important.

For constructive purposes the mechanical properties of these materials require to be known, together with the working strengths that may be adopted under various states of stress.

The terms 'elastic,' 'plastic' or 'ductile,' 'brittle,' and 'tough,' are used when referring to some of the mechanical properties of a material. It is said to be elastic if, after the application and release of a stress, it receives no strain—that is, no deformation or change of shape. Non-elastic or imperfectly elastic materials undergo a permanent change of form when subjected to stress; steel and wrought iron are nearly perfectly elastic for a certain range of stress, and imperfectly elastic for greater stresses. By 'plastic' or 'ductile' is meant that the material suffers a large deformation of shape under stress; 'brittle,' if it breaks under an increasing stress before any great deformation is caused; and 'tough,' if under an increasing stress it becomes plastic, and undergoes a large deformation of shape.

2. **Testing.**—To determine the mechanical properties of iron and steel, testing-machines are used, by means of which a series of tests can be carried out on the material under observation; of these the tests for tension, compression, shear, torsion, and bending are the more usual.

The diagram of fig. 1 is an autographic record of a tensile test carried out on a specimen of wrought iron, and will give an idea of the manner of the carrying out of such tests. The specimen having been placed in the machine, and connected to the recording apparatus, the stress is applied to the bar; and as the test proceeds the diagram is drawn. The value of the stress at any instant is given on the scale *OY*; and the strain, in this case the percentage elongation of the specimen compared with the original length, is measured along *OX*, the lines *OY*, *OX* being at right angles to one another. The curved line *Oabcd* gives the relation between the stress and strain at any instant; hence such diagrams are called stress-strain diagrams.

The straight line *Oa* shows that, for a certain range of the stress, the stress is proportional to the strain. On passing the point *a*, proportionality ceases; hence it is called the limit of elasticity for this particular kind of stress. At *b* the material elongates without increase of the load; it is called the yield point, the material being ductile; elongation of the specimen continues to take place, the strain becoming more defined for a small increase in the stress. At *c* the maximum stress, or breaking-load, is reached; and a local contraction, or drawing out of the bar, begins to be observable, finally breaking with the load at *d*, the intensity of the stress on the reduced section being greater than the 'breaking-stress' on the original area of the cross-section.

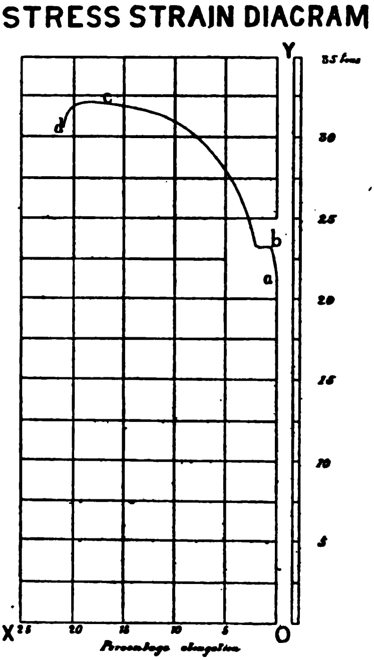

FIG. 1.

3. **Limits of Elasticity.**—For mild steel and wrought iron, the limits of elasticity are about two-thirds the breaking-strength: these limits depend greatly upon the history of the specimen. The rolling of the material raises the elastic limit for tension, but lowers it for compression. According to Bauschinger, the range of elasticity in a given material is fixed; and if the upper limit of elasticity is raised by rolling or wire-drawing for tension, the lower limit is necessarily lowered for compression. For a mild steel bar which has been carefully annealed, the limits of elasticity seem to be the same both for tension and

compression; on being subjected to rolling or wire-drawing, the limit is raised for tension, and lowered for compression. These results seem to be in accordance with what takes place in the testing-machine, as it is possible to raise the elastic limit considerably in a specimen by alternately loading and then partially unloading it during the test.

4. **Ultimate or Breaking-Strength.**—The breaking-strength, as determined in fig. 1, is usually called the statical breaking-load; it is not, however, the maximum load which could be applied to the bar. It has been shown experimentally that, if a bar be loaded slowly day by day, the statical breaking-load can be much exceeded, the result being due to an action analogous to wire-drawing in the specimen, its strength to compression, while in this condition, being lowered.

The strengths to tension and compression of mild steel are sensibly equal to one another; and the same may be said of wrought iron if the material is homogeneous and annealed. The rolling of the materials during the process of manufacture necessarily involves an analogous effect to wire-drawing, and they are sent out in this strained condition with the result that, when tested, the tensile strength is, as a rule, somewhat greater than the compressive strength. If such materials are annealed, the effects of the wire-drawing more or less disappear; and when tested, the tensile strength is less than in its previous condition, the elongation at the same time being increased.

For homogeneous materials, which may be assumed to be isotropic, such as mild steel and some kinds of annealed wrought iron, it is probable that the strength to tension, pure shear (torsion test), and compression are all equal to one another, and that the discordant results often obtained are due to the strained condition of the material. The testing of riveted joints shows that the shear strength of the rivet is less than its tensile strength; when it is remembered that the stress on the rivets is not one of pure shear, but complicated, with bending and local distortion, where the stress is applied, the results are not surprising.

5. **Elongation.**—The amount by which the bar is elongated or stretched during the test, compared with the original length, is used as a measure of the elongation or the permanent deformation, and therefore of the ductility or the plasticity of the material. The contraction of the area of the cross-section, where the bar breaks, is also used as a measure of

the ductility. As a practical test, it is not so easily measured as the elongation.

6. **Mild Steel and Wrought Iron.**—In judging of the quality of mild steel and wrought iron for construction purposes, the breaking-strength and elongation are considered together. The average values of the strengths for tension, compression, and shear (shackle-test) for different kinds of plates, bars, and rivets, are given in the following table:—

BREAKING-STRENGTH IN TONS PER SQUARE INCH.

	Tension.	Compression.	Shear.	Elongation per cent.	Contraction of area per cent.
IRON—					
Plates with grain,	22	18	18	15	10–25
Plates across grain,	20			10–15	
Bars, . . .	22–25	18–20	18–20	15–20	15–35
Rivets, . .	22–25			20	25–40
STEEL—					
Plates, . .	26–32	26–32	21–25	20	30–50
Bars, . . .	26–32	26–32	21–25	20	
Rivets, . .	24–29	24–29	20–25	25	40–50

In addition to satisfying the tests given in the above table, some specifications require the material to undergo one or more bending tests—strips cut from the plates being required to stand bending through 180° to a radius of curvature of about 1½ times the thickness of the plate. These tests give an indication of the ductility of the material; their use is not so much insisted on at the present time for bridge-work, the tests for elongation and contraction of area taking their place.

7. **Rivets.**—For these a material with a large percentage of elongation and contraction of area and a comparatively low tensile strength is usually specified. The heating and subsequent closing of the rivet demand a material in which ductility is a well-marked characteristic.

8. **Wöhler's Researches.**—When a bar is subjected to alternating stresses continually applied, the breaking-strength is less than when the stress is gradually applied, and does not alternate. Herr Wöhler, after a laborious investigation on different materials under alternating stresses in tension, bending, torsion, and compression, determined approximately a relation between the stresses that would break a bar under various conditions of loading; and further experiments by Spangenberg, Bauschinger, and Baker have confirmed these results.

The results may be briefly summed up as follows:—

(1) A bar subjected to a direct stress has a statical breaking-strength of t.

(2) Then the same bar subjected to an alternating stress varying between 0 and $\frac{t}{2}$ will break with a stress of $\frac{t}{2}$.

(3) Or, subjected to an alternating stress varying between $+\frac{t}{3}$ and $-\frac{t}{3}$, will break with a stress of $\frac{t}{3}$, the + and − signs denoting compression and tension, respectively. In other words, the stresses that will break the bar are in the proportion of $1 : \frac{1}{2} : \frac{1}{3}$.

The stresses alternating between 0 and $\frac{t}{2}$ may be looked upon as being made up of a direct stress of $\frac{t}{4}$, and an alternating stress of $+\frac{t}{4}$ and $-\frac{t}{4}$. Applying Wöhler's results to this case, the breaking-stress would be a direct stress of $\frac{t}{4}$, together with $\frac{3t}{4} = t$; hence the results of the experiments on alternating stress between equal, +, and − values may be looked upon as confirming the experiments on alternating stresses varying between 0 and some ± value.

It is a matter of opinion as to whether the fatigue of materials is accountable for the above results, or whether they should be ascribed to dynamic action, or partly to both these effects; at present the results of the experiments are generally looked upon as indicating a gradual breakdown of the material, but until further experimental data are forthcoming, no decisive opinion can be given.

All cases of varying stress may be reduced to a direct stress, varying between equal, +, and − values; for instance, a bar of, say, one square inch in cross-section is subjected to stresses varying from a certain maximum S to a certain minimum S, both of the same kind; it would be subjected to exactly the same stresses if a direct stress of

$$\min S + \frac{\max S - \min S}{2}$$

were applied to it, together with equal alternating stresses of

$$\frac{\max S - \min S}{2}.$$

If this result is interpreted by Wöhler's law, the static breaking-stress (ult S) that would replace these stresses would be

$$\begin{aligned}\text{ult } S &= \min S + \frac{\max S - \min S}{2} + \frac{3(\max S - \min S)}{2}\\ &= 2\max S - \min S;\end{aligned}$$

and by attention to the signs of the stresses, this result is applicable to all cases.

If the min S equal the max S and is of the same sign, there is no variation of stress, and

$$\text{ult } S = \max S.$$

If the min $S = 0$, then

$$\text{ult } S = 2\max S.$$

If the min S is equal to the max S and of opposite sign, then

$$\text{ult } S = 3\max S.$$

Another expression of the above result is as follows:—

$$\max S = \frac{\dfrac{\text{ult } S}{2}}{1 - \frac{1}{2}\dfrac{\min S}{\max S}}, \qquad (1)$$

which is convenient for calculating purposes. By dividing the numerator into the denominator, and neglecting all terms after the second, the expression becomes

$$\max S = \frac{\text{ult}\, S}{2}\left(1 + \tfrac{1}{2}\frac{\min S}{\max S}\right), \tag{2}$$

which is of the form of the Weyranch-Launhardt formula, much used on the Continent and in America for bridge-design.

9. **Factor of Safety.**—The loads to which a structure will be subjected being known, and the ultimate or breaking-strength of the material, it is necessary to decide on the working-strength to be adopted. This is obtained by dividing the breaking-strength by a number called the 'factor of safety,' or, as it is often tersely described, 'factor of ignorance,' and is determined by practical considerations and experience; hence

$$\text{working-strength} = \frac{\text{breaking-strength}}{\text{factor of safety}}.$$

Previous to the researches on alternating stress, various values were adopted for the factor of safety, depending upon the class of structure and manner of loading, for bridge-work values of 4 to 5, and even greater, were not uncommon. The tendency of the present practice is to take a factor of safety of 3 for a steady load, and to determine the working-strength of the material from a consideration of the maximum and minimum stresses to which it will be subjected. Adopting the value 3 for the factor of safety, and 30 tons per square inch for the breaking-strength of steel, and 22·8 tons per square inch for iron, the formulæ (1) and (2) become:

Steel—

$$\text{working-strength} = \frac{5}{1 - \tfrac{1}{2}\dfrac{\min S}{\max S}}. \tag{1a}$$

Iron—

$$\text{working-strength} = \frac{3{\cdot}8}{1 - \tfrac{1}{2}\dfrac{\min S}{\max S}}.$$

Steel—

$$\text{working-strength}\ 5 = \left(1 + \tfrac{1}{2}\frac{\min S}{\max S}\right). \tag{2a}$$

Iron—

$$\text{working-strength} = 3{\cdot}8\left(1 + \tfrac{1}{2}\frac{\min S}{\max S}\right).$$

These formulæ are easily remembered, and cover all cases that arise, attention being paid to the change of sign when the stresses alternate between compression and tension. The formulæ of (1*a*) for steel and iron give a more correct interpretation of Wöhler's experiments than the Weyranch-Launhardt formulæ of (2*a*), and for this reason the formulæ of (1*a*) will be used throughout this work.

10. **Range of Stress.**—The maximum and minimum stresses to which a bridge will be subjected depend upon the dead and live loads, the dead load being that due to the weight of the superstructure and girders, and the live load that due to the passage of locomotives, trucks, or other moving loads across it. For short bridges the dead load is small compared with the live load, and the range of stress large; and for long bridges the dead load is large compared with the live load, and the range of stress small. Any rational method of design must make provision for this variation of stress, and it is only by using the results of Wöhler's researches, as embodied in the foregoing formulæ, for the working strengths of the material that this can be done. Unfortunately the Board of Trade rules have hampered progress in this respect, as they make no provision for the variation of stress, and only consider the working strength for the live and dead loads combined, being 5 tons per square inch for iron, and 6½ tons per square inch for steel.

BENDING MOMENTS AND SHEARING FORCES.

11. **Stresses and Forces.**—The problem of the design of engineering structures resolves itself into two parts. Firstly, the evaluation of the stresses called into action in the various members of the structure by the external forces; and secondly, the proportioning of the various members to withstand these stresses.

In the case of bridge-work, the external forces acting on the structure are the live loads which traverse the bridge; and the dead loads, made up of the structure itself, and the dead load which it carries. For the present these loads will be assumed and the stresses called into action by these assumed loads determined. In what follows, when referring to forces, it is to be understood that the word 'weight' or 'load' might express the meaning equally well; if, for instance, a bar is stressed by a force of one ton, then a weight or load of one ton hanging on to the bar would produce the same stress. As a rule, weights or loads refer to forces acting in the direction of gravity; and it is these weights or loads which call into action the various forces which stress the various members of the structure. The word 'stress' then implies the resistance offered by the members of the structure to the applied forces.

The units in which forces, weights, loads, and stresses will be measured are pounds and tons.

12. **The Equilibrium of Forces.**—Let three forces act in one plane and at a point: they will be in equilibrium if their magnitudes and directions are represented by the sides of a triangle taken in order.

The experimental proof of this proposition, known as the triangle of forces, admits of easy demonstration; it can also be shown to be true in a simple manner as follows:—

Draw the triangle ABC (fig. 2), and imagine it to be the plan of a triangular prism floating in water with the axis vertical, since it *does not move*, the forces acting on it are in equilibrium.

Consider a small horizontal slice of the solid of unit thickness, and let p' be the pressure per unit of area on the faces; then on the faces AB, AC, and BC the pressures are $p'c$, $p'b$, and $p'a$, respectively; or, in other words, the pressures are proportional to the sides of the triangle. Bisect *AB*, *BC*, and *AC*, and erect perpendiculars: they will meet at the point *O*. Imagine the forces *P*, *Q*, and *R* to act along these lines. Let $P = p'c$, $Q = p'b$, and $R = p'a$; then $P : Q : R :: c : b : a$, that is, the three forces *P*, *Q*, *R* meeting at the point are proportional to the sides of the triangle drawn at right angles to the forces,

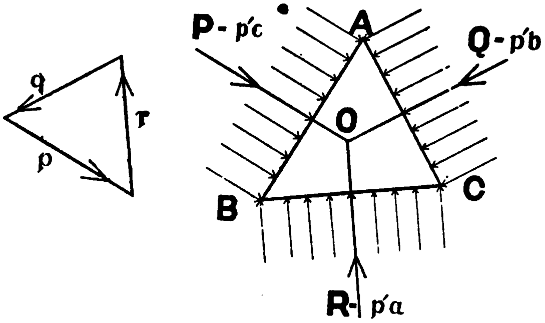

Fig. 2.

and since the forces $p'a$, $p'b$, and $p'c$ were in equilibrium, so also are the forces *P*, *Q*, and *R*. Draw another triangle with its sides *p*, *q*, and *r* perpendicular to *c*, *b*, and *a*; the triangles will be similar and the sides proportional. Hence $P : Q : R :: p : q : r$; but *p*, *q*, and *r* are parallel to *P*, *Q*, and *R*, respectively: hence the proposition is proved.

13. **Polygon of Forces.**—If any number of forces act at a point, they will be in equilibrium if their magnitudes and directions are represented by the sides of a polygon taken in order.

This proposition follows at once from the triangle of forces. Let *P*, *Q*, *R*, *S*, *T*, *U* be the forces acting at the point (fig. 3), and draw the sides of the polygon *pqrstu* in order, representing the magnitude and direction of the forces.

Consider the two forces *P* and *Q* acting at the point *O*, and the two sides *p*,*q*, of the polygon, join their extremities by the straight line *a*; a third force *A* introduced at the point *O* represented in magnitude and direction by *a* would produce equilibrium with the forces *P* and *Q*, their magnitude and directions being represented by *p*, *q*, and *a*; if now the force *A* be reversed in direction at the point, its effect would be the same as that of the forces *P* and *Q*, and could replace them; in other words, the force *A* is the resultant of *P* and *Q*. Similarly the force *B* could replace the forces *A* and *R*, and the force *C* could replace the forces *B* and *S*, and there would be left the force *T*, *U*, and *C*, acting at the point and in equilibrium, their magnitudes and directions being represented by *t*, *u*, and *c*.

The polygon *pqrstu* is called the force-polygon of the forces acting at the point *O*. The condition, then, for the equilibrium of a system of plane-forces acting at a point is that the force-polygon must close. This is equivalent to the statement that the algebraic summation of the forces on any two right lines = 0; or, as usually expressed, $\Sigma F = 0$.

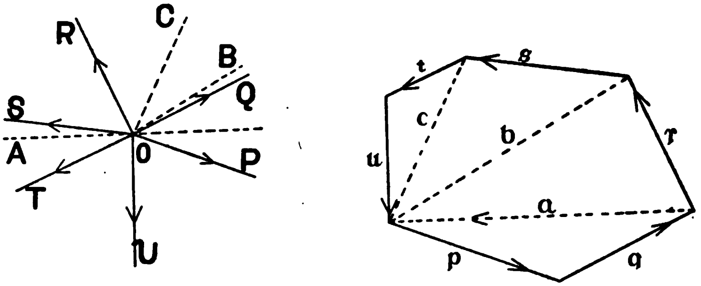

FIG. 3.

If the force-polygon does not close, the system reduces to a resultant which is represented in magnitude and direction by the closing-line of the force-polygon acting in the opposite sense to that of the forces round the polygon.

14. **Forces acting in a Plane.**—Suppose the forces *P*, *Q*, *R*, *S*, *T*, and *U* given in direction and magnitude in fig. 4 but not acting at a point, but anywhere in the plane of the paper. What are the conditions of equilibrium?

Draw, as before, the force-polygon *pqrstu*, and take any point *O* and join it to the extremities of all the sides of the polygon. Select any point on the force *P*, and draw from it the straight lines *A* and *B* parallel to the sides of the force-polygon *a* and *b*. Produce the line *A* to meet on the force *U*, and the line *B* to meet on the force *Q*; there are three forces represented in magnitude and direction by *p*, *a*, and *b* acting at a point: hence they are in equilibrium; of these, *P* is the given force, and the other two forces act along *A* and *B*.

From where the line *B* intersects the force *Q*, draw the line *C* parallel to the line *c* of the force-polygon; similarly, from the intersection of the line *C* on the force *R*, draw the line *D* parallel to the line *d* of the force-polygon, and repeat the construction for the lines *E* and *F*; if the forces are in equilibrium, the line *F* and the line *A* will intersect on the force *U*. Now the force *P* was shown to be in equilibrium with the forces acting along the lines *A* and *B*; also similar reasoning applies to the lines intersecting on the other forces: hence the whole system is in equilibrium. The linkage *A*, *B*, *C*, *D*, *E*, and *F* is called the link-polygon, meaning a system of links which would be in equilibrium under the action of the given forces. The point *O* is called the pole of the force-polygon, since it can be taken anywhere. A number of linkages can be drawn, all of which will be in equilibrium under the action of the given forces. Two conditions require to be fulfilled for a system of plane forces to be in equilibrium, viz.:—

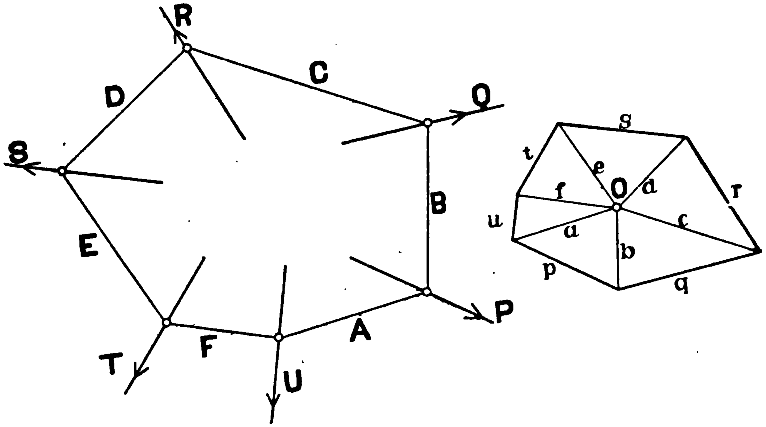

Fig. 4.

(1) The force-polygon must close; that is, the algebraic summation of the forces acting on any two right lines = 0, or $\Sigma F = 0$.

(2) The link-polygon must close; that is, the algebraic summation of the moments on any two right lines = 0, or $\Sigma M = 0$.

If the force-polygon does not close, the link-polygon closes, and the system reduces to a resultant.

If the force-polygon closes, and the link-polygon does not close, the system reduces to a couple.

15. **Bending-Moment Diagrams.**—The moment of a force is defined by the product of the force by the perpendicular distance of the force from the point.

For the problems required in bridge-work the external forces acting on the structure are, as a rule, parallel to the direction of gravity, and the perpendicular distances to these forces are measured from some point in and along the span of the bridge.

In fig. 5, five weights W_1, W_2, W_3, W_4, W_5 are shown acting between P and Q at the extremities of the span; it is required to determine the magnitude of the supporting forces P and Q, and the bending moment at any point in the span.

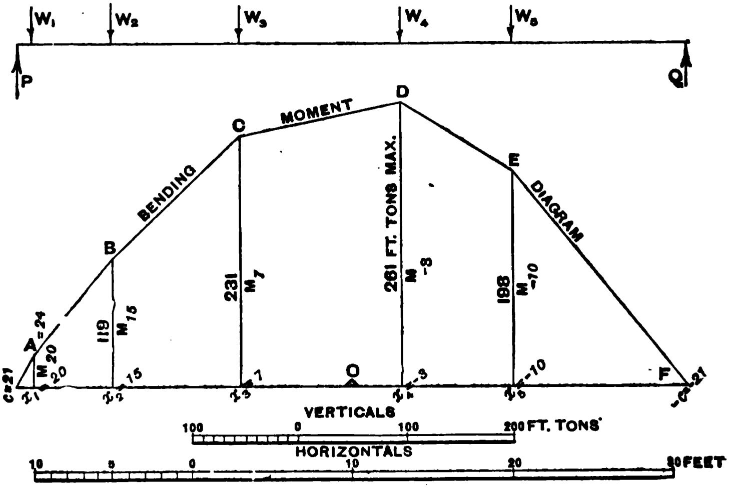

Fig. 5.

Let x_1, x_2, x_3, x_4, x_5 represent the distances of the weights W_1, W_2, W_3, W_4, W_5, respectively, measured from the origin O, the centre of the span; take moments round P: then, since the system is in equilibrium, $\Sigma M = 0$, that is,

$$W_1(c-x_1) + W_2(c-x_2) + W_3(c-x_3) + W_4(c-x_4) + W_5(c-x_5) - Q2c = 0:$$

this determines Q; since the forces are parallel, the sum of

$$P + Q = W_1 + W_2 + W_3 + W_4 + W_5,$$

that is $\Sigma F = 0$: hence P is known.

The bending moment at any point of the span is defined as the algebraic summation of the moments of the external forces on either side of the point; at x_3 its value is

$$P(c - x_3) - W_1(x_1 - x_3) - W_2(x_2 - x_3) = M.$$

Similar equations giving the values for any other point.

By computing the values of the bending moments at several points, and marking off perpendiculars to the span at these points proportional to the bending moments, and joining the extremities of the perpendiculars together, a figure is obtained which is called a bending-moment diagram.

In fig. 5 the values of the weights and the distances measured along the span are all shown together with the bending moments under each weight.

16. **Scales.**—Since a moment is the product of force multiplied by distance, the units in which it is measured must contain units of force and of distance. Usually the units used for bridge-work are the inch-pound, inch-ton, foot-pound, and foot-ton for moments, and feet and inches for distances.

In making a drawing of a bending-moment diagram scales are necessary for plotting bending moments and distances; the scales being drawn on the paper, the values can be scaled off directly with the compasses from the diagram. On fig. 5 two scales are shown; and the drawing being to scale, the values marked on the figure can be verified.

17. **Culmann's Theorem.**—The bending-moment diagram of fig. 5, which was determined by the calculation of the bending moments at various points of the span, can also be derived directly from the construction given for the equilibrium of forces acting in a plane: § 14, the construction required, being a particular case of the general solution. Thus, let the weights be of the same magnitude and positions as given on fig. 5; it is required to determine a linkage on which the forces act, and the reactions P and Q when the system is in equilibrium.

Draw the lines W_1, W_2, W_3, W_4, W_5 in order, representing the magnitude and directions of the sides of the force-polygon to scale, fig. 6. In this case, the force-polygon is made up of straight lines; and since the system is in equilibrium, the force-polygon must close—close it by drawing the line fa. Choose any pole O', and join it to $abcdef$ by vectors, as shown. From H', any point on HH', draw $H'A'$ parallel to $O'a$ and meeting W_1 at A'; draw $A'B'$, $B'C'$, $C'D'$, $D'E'$, and $E'K'$ parallel respectively to $O'b$, $O'c$, $O'd$, $O'e$, $O'f$, and join $H'K'$: it is the closing line of the link-polygon. Draw *O'g* parallel to *H'K'*, meeting *af*, the closing line of the force-polygon, in *g*. Then $fg = Q$, and $ga = P$, the force required to produce equilibrium. The linkage, together with the forces acting on it, is in equilibrium, because the three forces acting at every point are in equilibrium, being proportional to the sides of a closed triangle of the force-polygon.

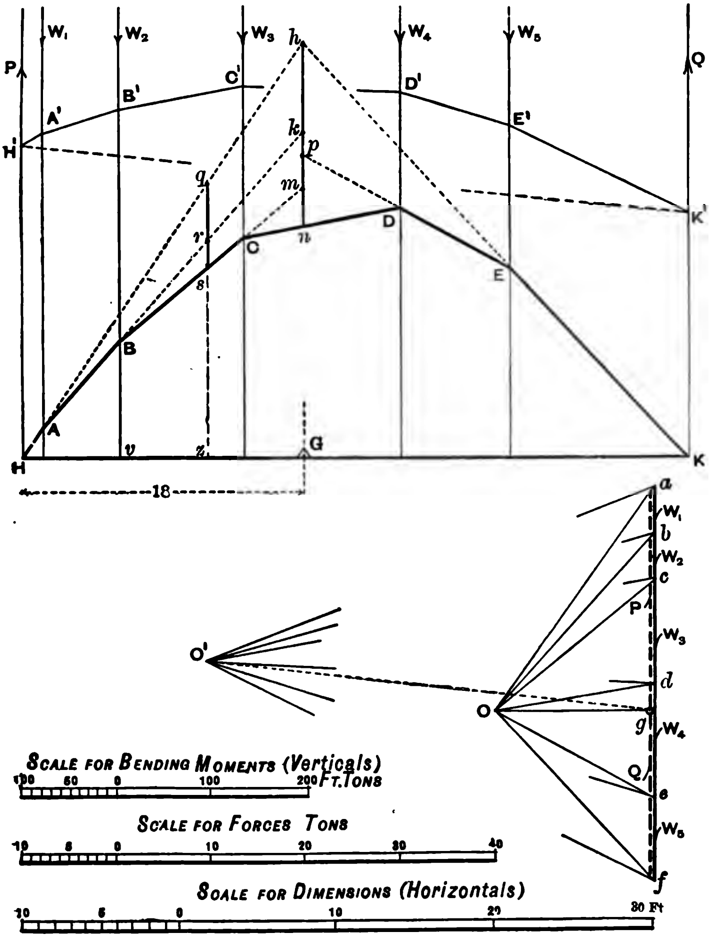

Fig. 6.

It is convenient to draw a linkage on which the forces act in which the closing line of the link-polygon is horizontal: the base line of the diagram is then parallel with the span. To do this, take a new pole O on a horizontal through g of the force-polygon, and, starting from the point H on the span, repeat the construction for $HABCDEFK$. The link-polygons so drawn are bending-moment diagrams for the given forces. Confining our attention to the second polygon, this may be shown as follows:—

Produce the links HA, EK to meet at h, and from h drop the perpendicular hG to the base; produce the links AB, BC, DE to meet this perpendicular in k, m, and p respectively, the link CD cutting it at n. Draw any intercept such as $zsrq$. The triangle Bsr is similar to the triangle Obc of the force-polygon; hence

$$\frac{rs}{bc}=\frac{Bs}{Oc}; \quad \text{also} \quad \frac{Bs}{Oc}=\frac{vz}{Og};$$

therefore,

$$Og \, . \, rs = bc \, . \, vz, \quad \text{and} \quad bc = W_2$$

$$= W_2 vz = \text{moment of } W_2 \text{ about } z,$$

rs and vz being dimensions, and Og and W_2 forces.

Again,

$$Og \, . \, zq = ag \, . \, Hz = P \, . \, Hz = \text{moment of } P \text{ about } z.$$

And similarly,

$$Og \, . \, qr = \text{moment of } W_1 \text{ about } z.$$

Summing up these moments, and noting that P acts in the opposite direction to W_1 and W_2, the expression is

$$Og(zq - rs - qr) = \text{moment of } P - \text{moment of } W_2 - \text{moment of } W_1;$$

or,

$$Og \, . \, sz = \Sigma M;$$

that is, the intercept sz of the link-polygon is proportional to the bending moment at the point Og, being constant, and of the same value no matter where the intercept is taken. Hence the link-polygon is also the bending-moment diagram for the given forces. This is Culmann's theorem.

The intercept Hg, found by producing the links HA, EK, gives the direction of the resultant of all the forces, or, what is the same thing, the centre of gravity of the weights. If, instead of finding the forces P and Q to produce equilibrium, one force equal to $P + Q$ had been applied at h to the linkage $ABCDEh$, it would be in equilibrium under the action of the given weights. Thus, the moments about h are

$$nm + mk + kh - hp - pn = 0; \quad \text{that is,} \quad \Sigma M = 0.$$

Also, from the force-polygon,

$$W_1 + W_2 + W_3 + W_4 + W_5 - (P + Q); \quad \text{that is,} \quad \Sigma F = 0.$$

By constructing a scale of bending moments, that is, in this case, a scale in which the dimensions have been multiplied by the force Og, the values of the bending moments can be read off directly from the diagram. Fig. 6 is drawn to scale; and the scales for the bending moments, forces, and dimensions are given thereon.

18. **Shearing-Force Diagram.**—The shearing force at any point of the span is the algebraic sum of the external forces to either side of the point.

On fig. 7, the same weights, together with the known reactions, are given as on figs. 5 and 6; and since they are in equilibrium, $\Sigma F = 0$; that is,

$$-P + W_1 + W_2 + W_3 + W_4 + W_5 - Q = 0.$$

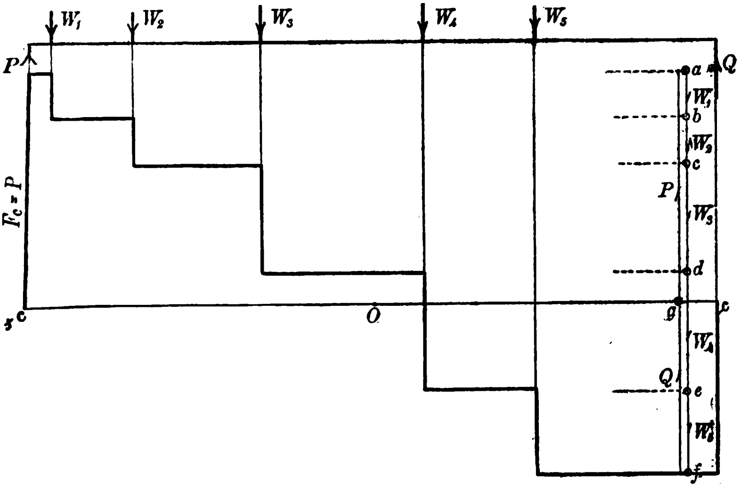

Fig. 7.

The shearing force at any point between, say, W_3 and W_4, will be

$$-P + W_1 + W_2 + W_3 = -W_4 - W_5 + Q.$$

To determine the shearing-force diagram.—Draw the force-polygon $W_1 W_2 W_3 W_4 W_5 Q$ and P to scale in any convenient position; through g draw a horizontal line, and produce the lines of action of P and Q. Similarly, through a draw a horizontal line to cut the lines of action of W_1 and P, and repeat the construction for the points b, c, d, e, f; the stepped figure on the horizontal line thus determined is the shearing-force diagram. The ordinates to this diagram at any point on the span give the algebraical sum of the forces at that point.

19. **Load Diagrams.**—On figs. 5 and 7 the positions of the weights on the span are shown by arrows; the base line, together with the arrows, constitutes what is called a load-diagram for the concentrated loads situated at the various points of the span. When the span is loaded all over with a uniform load, the diagram becomes a rectangle as shown on fig. 8, the height of the diagram to scale corresponding to the load in tons or lb. per foot on the span. The load, shearing-force, and bending-moment diagrams form the starting-point in all problems connected with the design of bridges, and should always be considered together. For brevity, they will hereafter be referred to as the L., the S. F., and B. M. diagrams.

20. **Beam uniformly Loaded.**—The L. diagram in this case is a rectangle shown on fig. 8. Let w be the intensity of the load in lb. per foot, and put the length of the span $= 2c$, then the total load $W = 2wc$. To find the bending moment at a point distant x from the centre O. Consider the load-area standing upon the span to the left of the point: it is a rectangle of length $(c - x)$ feet, and weight $w(c - x)$ lb. This weight, acting at its centre of gravity, that is, at the middle point of the segment $(c - x)$, gives the same bending moment about x as that of the actual distribution. Equating the moment about x, the expression is

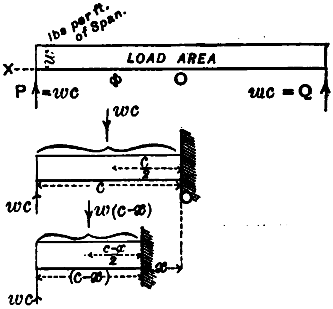

FIG. 8.

$$M_x = P(c - x) - w(c - x)\left(\frac{c - x}{2}\right).$$

Since the load is uniformly distributed, the reactions P and Q are each equal to wc: hence

$$M_x = \frac{w}{2}(c^2 - x^2) \quad \text{if} \quad x = \pm c, \quad M_c = 0, \quad \text{that is at the ends of the span;}$$

$$\text{if} \quad x = 0, \quad M_0 = \frac{wc^2}{2} = \frac{Wl}{8}, \quad \text{at the centre of the span.}$$

The curve of the B. M. diagram is a parabola; this is shown on fig. 9. To draw the diagram, mark off $\frac{1}{8}WL$ on a perpendicular to the base at its centre on some convenient scale, and construct the curve passing through the extremity of the perpendicular and the two ends of the span, using some of the known geometrical constructions for the curve, or calculating the values directly for a number of points, and plotting the curve through the extremities of the calculated ordinates.

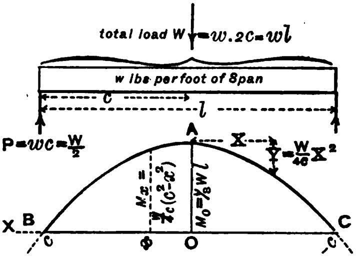

Fig. 9.

The S. F. diagram is determined thus (fig. 10): mark off the values of P and Q to scale perpendicular to the base line, and join the extremities by a straight line: it is the diagram required.

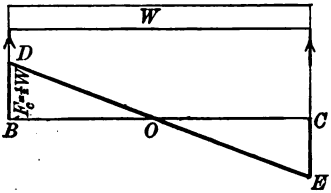

Fig. 10.

21. **The Relation between the L., S. F., and B. M. Diagrams.**—The shearing force at any point of the span has already been defined as the algebraic sum of the forces to either side of the point, that is, the shearing force = ΣF. This is true for all conditions of load, whether uniform, varying, or concentrated.

If the load on the span can be expressed as a function of the distance measured along the span, the shearing force at any point can be determined by the use of the calculus, the integration of the expression for the load up to any point giving the shearing force at that point. Thus, for a uniform load on the span, the shearing force at the centre = 0 (fig. 10); and for values of x, measured from O, the centre of the span, the shearing force

$= wx$; that is, it is proportional to the area of the L. diagram. The S. F. diagram on fig. 10 shows this, being made up of two triangles; the intercepts perpendicular to the span vary in direct proportion to their distance from the centre. Now the relation here established for the S. F. diagram for the uniform load is true for any distribution of loading; hence the value of the shearing force at any point can be expressed as $F = \int wdx$, where w is either constant or some variable depending on x.

For the uniform load $w =$ constant; hence

$$F = w\int dx = wx + C.$$

Now when

$$x = 0, \quad F = 0; \quad \therefore C = 0,$$

and no constant is required. This is the equation of a straight line, and, if plotted, would give a S. F. diagram similar to that of fig. 10.

The B. M. diagram is related in a similar way to the S. F. diagram that the S. F. diagram is to the L. diagram. The bending moment at any point is proportional to the area of the S. F. diagram up to that point. The S. F. diagram on fig. 7, and the B. M. diagram on figs. 5 and 6, clearly show this. The horizontal lines through g and f (fig. 7), together with the lines produced through W_5 and Q, enclose a rectangle; and the area of this rectangle is proportional to the intercept on and through E of the B. M. diagram of fig. 5. If a point were taken midway between W_5 and Q, the area on the S. F. diagram would be halved; so also would the intercept on the B. M. diagram. Similarly, the intercept through D on the B. M. diagram is proportional to the areas of the rectangles enclosed between W_4W_5 and W_5Q of the S. F. diagram. For the B. M. and S. F. diagrams for the uniform load (figs. 9 and 10) the same relation can be shown by dividing the S. F. diagram into a number of slices perpendicular to the base; and marking off intercepts on the B. M. diagram proportional to the sum of these areas at any point, the curve obtained for the B.M. diagram will be a parabola. The above relation, expressed by the use of the calculus, is

$$M = \int Fdx, \quad \text{or} \quad \frac{dM}{dx} = F;$$

the rate of increase of the bending moment equals the shearing force, but $F = \int wdx$, hence $M = \iint wdx^2$: for the case of the uniform load $F = wx$, hence

$$M = \int Fdx = w\int xdx = \frac{wx^2}{2} + C.$$

Now, when

$$x = c, \quad M = 0; \quad \therefore \ C = -\frac{wc^2}{2}, \text{ and } -M = \frac{w}{2}(c^2 - x^2),$$

the equation of a parabola.

22. **Combined Diagrams.**—If two B. M. or S. F. diagrams, drawn to the same scale for different loads and for the same span, are given, then the B. M. or S. F. diagrams for the combined loads can be obtained by the addition of the ordinates of the given B. M. and S. F. diagrams. On fig. 11 the B. M. diagrams for a load at the centre of the span, and for a uniform load over the whole span, are shown, and the combined B. M. diagram for the combined loads obtained by the addition of the ordinates.

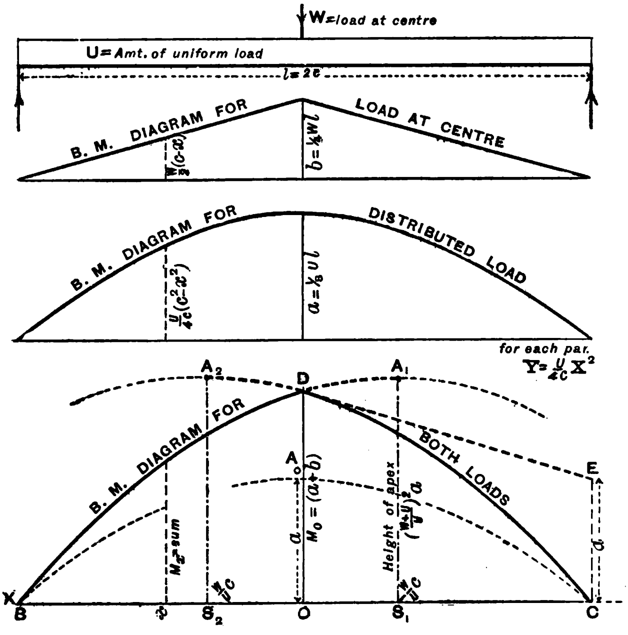

Fig. 11.

For the concentrated load, the bending moment at any point $= \frac{W}{2}(c - x)$, and at the centre $= \frac{1}{4}Wl$; for the uniform load, the bending moment at any point $= \frac{U}{4c}(c^2 - x^2)$, and the centre $= \frac{1}{8}Ul$; W being the concentrated load on the span, and U the uniform load.

For the combined B. M. diagram, the bending moment at any point $= \frac{U}{4c}(c^2 - x^2) + \frac{W}{2}(c - x)$, and at the centre $= (\frac{1}{8}U + \frac{1}{4}W)l$. These values are shown on the figure, and admit of easy verification.

The combined B. M. diagram is made up of two arcs of the same parabola as that for the uniform load, the vertices of the parabolas A_1, A_2 being distant $\frac{W}{U}c$ from the centre of the span.

The diagram can be readily drawn by marking off the sum of the bending moments at the centre of the span, and inscribing the parabolic arc in the triangle DEC, and using the same construction for the other side of the diagram.

23. **S. F. and B. M. Diagrams for Moving Loads.**—The S. F. and B. M. diagrams already given relate only to fixed loads, or, as usually called, dead loads. For most bridges there is, in addition to this load, a moving or live load to be considered; it is the S. F. and B. M. diagrams for these that will now be determined.

Moving loads refer, as the name implies, to loads which traverse the span of the bridge, such as locomotives, trucks, and the like. A uniform moving load coming on at one end of the span, covering part or all over the span, and passing off at the other end, is spoken of as an advancing load; for instance, a train of trucks nearly fulfils this condition.

A load concentrated at a point, such as a loaded wheel, is spoken of as a rolling load, and several rolling loads, rigidly grouped together as in a locomotive, as a travelling load system.

The B. M. and S. F. diagrams for a rolling load.—On fig. 12 the rolling load R is shown distant x from O, the centre of the span: calculate the reactions P and Q; then the moment at x is

$$M_x = P(c - x) = Q(c + x).$$

Now if the load R moves either to the right or left of the point, the value of the moment at x, M_x decreases; hence the bending moment at any point is greatest when the load is over the point. Substitute for Q in terms of R; thus

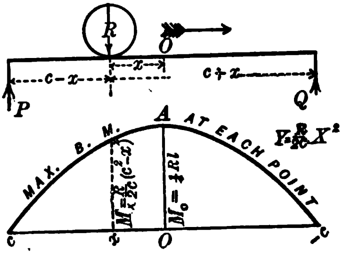

Fig. 12.

$$Q \,.\, 2c = R(c - x):$$

hence $$M_x = \frac{R}{2c}(c^2 - x^2);$$

and this is the equation for the maximum bending moments at any point. The maximum of these maxima occurs when $x = 0$: then $M_{\text{max.}} = \frac{1}{4}Rl$, and the B.M. diagram is a parabola.

For the S. F. diagram (fig. 13) place the load R in any position on the span as shown, and calculate the reactions P and Q; these give the value of the positive and negative shearing force at the point; and they are also their maximum values. If R moves to the left of the point, the value of Q decreases: hence the negative shearing force decreases. Again, if R moves to the right, the value of P decreases, and the positive shearing force decreases: hence the value of the shearing forces is a maximum when the load is over the point.

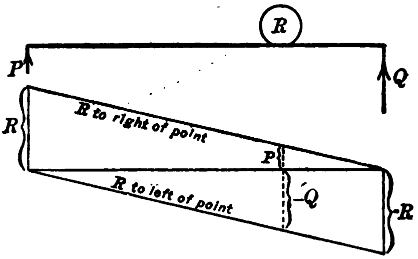

Fig. 13.

To draw the diagram, drop perpendiculars to the extremities of the base line, and mark off the value of R above and below the line; draw a pair of parallel lines passing from the ends of the base to the ends of the perpendiculars. The positive and negative shearing forces are given by the intercepts above and below the base line; and their sum is obviously always equal to the load.

24. **Advancing Load equal in Length to the Span.**—

The bending-moment diagram will be the same as that for a uniform load of the same intensity as the advancing load distributed over the span, the bending moment at any point being greater when the span is loaded all over than when partly loaded; thus consider the B. M. diagram drawn for the span partly loaded, and another B. M. diagram drawn for the load distributed on the other part of the span, the combined diagram would be larger than either of the B. M. diagrams separately: hence the bending moment at any point is a maximum when the span is loaded all over; and the case resolves itself into that of a beam uniformly loaded (§ 20). If U is the total load over the span, then the bending moment at the centre will be $\frac{1}{8}Ul$; it will be noted that this value is one-half of what it would have been had the load been a rolling one.

To determine the S. F. diagram.—Suppose the span to be

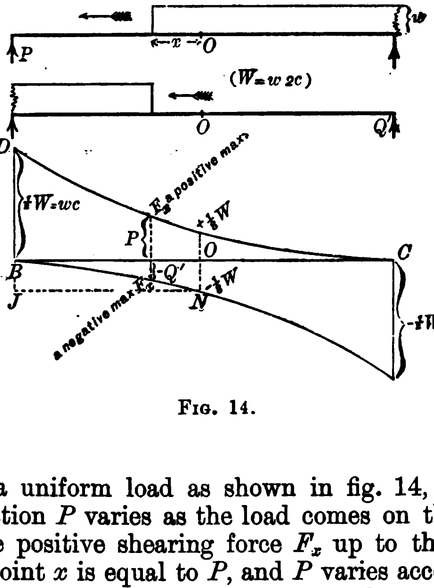

Fig. 14.

loaded with a uniform load as shown in fig. 14, and consider how the reaction P varies as the load comes on the span from the right, the positive shearing force F_x up to the end of the load at the point x is equal to P, and P varies according to the position of x, that is, the extent of the load on the span. If the load moves to the right of the span, the value of P or F_x decreases at the point x. If it moves to the left and partly passes, P increases, and so also does the load by the amount it goes beyond x. Now the increase of P is less than the increase of the load, and the shearing force is again less; hence the shearing force is a maximum when the front of the load is at the point. A similar demonstration applies to the negative shearing force.

To draw the diagram, calculate the value of P, that is, F_x for any value of x: thus—

$$P \,.\, 2c = w(c+x)\left(\frac{c+x}{2}\right); \quad \therefore\ P \text{ or } F_x = \frac{w}{4c}(c+x)^2,$$

a parabola; if $x = -c$, $F_x = 0$, the curve starts from the end of the span; if $x = +c$, $F_x = wc$, the value of the ordinate to the curve at the other end of the span as shown on fig. 14. The curve is reversed for the negative shearing force, the diagram being symmetrical.

25. **Combined Live and Dead Loads.**—The B. M. and S. F. diagrams for the dead and live loads have now to be combined together; for the simple cases so far dealt with, this can be readily done. Consider the span to be loaded with a uniform dead load and a uniform live load. The B. M. diagram is a parabola in each case; and the combined B. M. diagram will be the sum of the ordinates of the separate diagrams and another parabola. If w and u are the intensities of the dead and live loads, respectively, the equation to the curve is

$$M_x = \frac{w+u}{2}(c^2 - x^2).$$

The S. F. diagram will be determined from the algebraic sum of the ordinates of fig. 10 for the dead load, and fig. 14 for the live load. The diagram obtained is shown in fig. 15. *BDOCE*

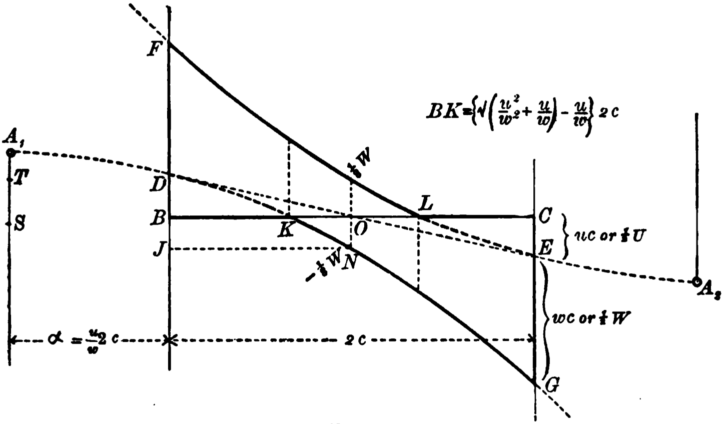
F
$BK = \left\{\sqrt{\left(\frac{u^2}{w^2} + \frac{u}{w}\right)} - \frac{u}{w}\right\} 2c$
A₁
T
S
D
B
K
O
L
C
E
J
N
$\frac{1}{2}W$
$-\frac{1}{2}W$
uc or $\frac{1}{2}U$
wc or $\frac{1}{2}W$
A_2
$\alpha = \frac{u}{w} 2c$
$2c$
G

FIG. 15.

is the S. F. diagram for the dead load; and the triangles *DFE*, *DEG*, with the inscribed parabolic arcs, are the S. F. diagram for the moving load of fig. 14 after distortion by the addition of the ordinates of the separate diagrams. The parabolic arc *FLE* cuts the base *BC* at the point *L*. The positive shearing force at that point of the S. F. diagram for the live load equals the negative shearing force of the S. F. diagram for the dead load; and therefore the shearing force at *L* varies between *O* and the value of the ordinate through *L*. For the portion of the span *LC*, the shearing force is always negative; and similarly, for the portion *KB*, the shearing force is always positive. For the part of the span between *LK*, the shearing force varies from positive to negative values, varying at the centre from $\frac{1}{8}W$ to $-\frac{1}{8}W$. It is this portion of the span which requires counterbracing. The stresses on the members change in sign during the passage of the live load, owing to the change in sign of the shearing force. The majority of the problems connected with the design of plate and triangulated girders depend for their solution on the use of the B. M. and S. F. diagrams just described; and the reasoning which underlies their construction must be carefully studied.

26. **Equivalent Load Diagram.**—It is convenient to determine what distribution of a dead load will give the shearing-force diagram of fig. 15; this is shown on fig. 16, and is made up of a uniform dead load of $\frac{1}{2}$ the intensity of the moving load, together with the dead load itself; a concentrated load of $\frac{1}{4}$ the moving load placed at the centre of the span; and a pair of wedge-shaped loads of $\frac{1}{8}$ the total moving load, varying from nothing at the centre to $\frac{1}{4}$ the intensity of the moving load at the ends of the span. The proof of this can easily be derived by geometry from the S. F. diagram.

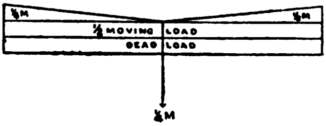

FIG. 16.

THE THEORY OF BENDING AND DISTRIBUTION OF STRESS.

27. **Design of Girders.**—The problem of the design of girders divides itself into two parts: first, the determination of the bending moments and shearing forces at every section of the girder due to the live and dead loads; and secondly, the proportioning of the various parts of the girder to resist these bending moments and shearing forces. The first part of the problem deals with the determination of the B. M., S. F., and L. diagrams; the investigation underlying the second part will now be given.

28. **Hooke's Law.**—Reference has already been made to the elastic properties of materials, and also to the safe working stress to be adopted under certain conditions of loading. For the design of girders, it is assumed that the strains produced in any part are always within the elastic limits of the material, and therefore that Hooke's law, viz. that the stress is proportional to the strain, holds under all conditions of loading.

Let a bar of length L be loaded with a weight P, as in fig. 17: then, if A is the sectional area, the load per unit area $p = \frac{P}{A}$. Under the action of the load, the bar will increase in length by an amount l; this quantity divided by the original length L is called the longitudinal strain, and denoted by the letter e; hence, since the material is elastic, it follows from Hooke's law that p varies as e or $p = Ee$, where E is a constant quantity, substituting $\frac{l}{L}$ for e, and rewriting the equation in the form $\frac{p}{E} = \frac{l}{L}$; then, if $p = E$, $l = L$: hence, the hypothetical load that would stretch a bar by an amount equal to its own length is equal to E; this constant is called Young's modulus of elasticity.

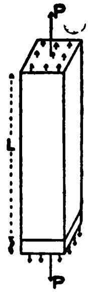

FIG. 17.

29. **Application to Beams.**—To apply Hooke's law to beams, consider a beam of the section shown on fig. 18, and made up of a series of laminæ. If two equal and opposite

couples are applied to such a beam, it will bend into an arc of a circle, the upper layers being lengthened or stretched, and the lower shortened or compressed. There will be one layer which undergoes no change in length; this layer is called the neutral axis or plane of the beam. Let R be its radius. The top layer will be increased in length l'; and, if L is the original length of the layers before bending, then the strain $\frac{l'}{L} = e$. Since in circles the arcs are proportional to the radii, the following relation obtains, from fig. 18:

$$\frac{R+y}{R} = \frac{L+l'}{L} \text{ or } \frac{y'}{R} = \frac{l'}{L}:$$

but from Hooke's law $\frac{l'}{L} = \frac{p}{E}$; therefore $\frac{p}{E} = \frac{y'}{R}$, that is, the pull on the layer varies directly as its distance from the neutral axis, and this is true for any value of y. Hence the pull or thrust on any layer $= \frac{Ey}{R}$ or $\frac{p}{E} = \frac{y}{R}$.

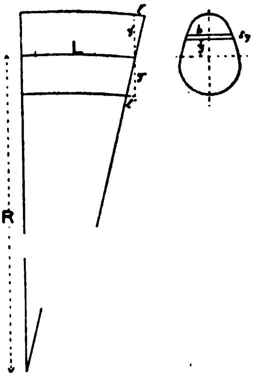

FIG. 18.

30. **Determination of the Neutral Axis.**—The forces acting on the section of the beam (fig. 18) are two equal and opposite couples. Consider one couple: it is made up of two equal and opposite forces distributed over the section, and their algebraic sum is equal to 0. Now these forces are distributed above and below the neutral axis. Consider one layer of breadth b and thickness δy; there is acting on it a pull of p per unit area: hence the total pull on the layer $= pb\delta y$.

If the beam is divided into a number of layers, the algebraic sum of the pulls and thrusts over the whole section $= 0$; or, what is the same thing, $\int bydy = 0$. From § 29, $p = \frac{Ey}{R}$, substituting the value of $\int pbdy$, becomes $\frac{E}{R} \int bydy = 0$. This summation has a definite name, and is called the geometrical moment, meaning the algebraic sum of each elementary area into its distance from some axis. In this case it is to be zero. This can only be true if the neutral axis passes through the centre of gravity of the section; for it is the same thing as saying that the moment of the area about the neutral axis is to be zero.

31. **Determination of the Moment of Resistance.**—On applying two equal and opposite couples to the beam of fig. 18, it bends into an arc of a circle; and the internal stress in the material resists the bending moment of these couples. To determine the stress set up in the different layers, consider the layer shown on the section of fig. 18: there is acting on it a pull of $pb\delta y = \frac{E}{R} by \delta y$. Now this pull acts with a leverage of y round the neutral axis: hence, the moment of the forces acting on the elementary area $= \frac{E}{R} by^2 \delta y$. If the algebraic sum of all the moments be taken over the whole area, then the bending moment $M = \frac{E}{R} \int by^2 dy$; this sum has a definite name, and is called the Moment of Inertia, and represented by I, meaning the summation of each little elementary layer multiplied by the square of its distance from some axis, in this case the neutral axis. Substituting I for $\int by^2 dy$, the relation $M = \frac{E}{R} I$ is obtained, but $\frac{p}{y} = \frac{E}{R}$ from § 29; therefore $\frac{p}{y} = \frac{M}{I} = \frac{E}{R}$, and from which $M = \frac{pI}{y}$. Now M is the moment of the external forces which is resisted by the moment of the internal stresses of the beam, that is, $\frac{pI}{y}$, where p is the intensity of the pull or thrust on the layers of the beam, I the moment of inertia of the section, and y the distance of the layers from the neutral axis. This quantity is called the moment of resistance, and is continually required for the design of beams.

32. **Moment of Inertia.**—The moment of inertia has been defined as the algebraical summation of each little elementary area of the section into the square of its distance from some axis. If the neutral axis is taken for any particular section, then the moment of inertia is a minimum for that section. The value of the moment of inertia round any parallel axis is equal to the moment of inertia round the neutral axis, added to the area of the section into the square of the distance between the two axes. Thus (fig. 19) let Y be the distance

between the two axes, and y the distance from the neutral axis to any elementary area; then, I for the neutral axis $= \int by^2dy$. Now the moment of inertia round any other axis will be given by the expression

$$I = \int b(y+Y)^2\,dy = \int by^2dy + 2Y\int bydy + Y^2\int bdy,$$

Now $\int by^2dy =$ the moment of inertia round the neutral axis, and $\int bydy = 0$, since the axis passes through the centre of gravity of the section, and $\int bdy = A$ the area of the section: hence,

$$I_0 = I + Ay^2.$$

If two sections are given of which the moments of inertia, I_1 and I_2 are known, referred to the same axis; then the moment of inertia

$$I_0 = I_1 + I_2.$$

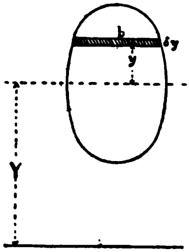

FIG. 19.

Similarly for the geometrical moments. If the geometrical moments for two or more sections referred to the same axis are known, then the geometrical moment for the whole section is the sum of the geometrical moments for the separate sections added together.

For a great many sections used in bridge-design, it is necessary to split up the section into a number of areas, and to determine the geometrical moments and the moments of inertia of each of the areas separately; the sum of these giving the required values for the whole section.

The moment of inertia of comparatively few sections can be expressed in a concise form. It is therefore preferable, when the moment of inertia for a particular section is required, to proceed by a method which gives a close approximation rather than by deducing complicated expressions for it mathematically.

Let a rectangle be given, fig. 20, Y and y representing the distance of the sides parallel to the axis XX, and b the breadth; then the area of the rectangle

$$A = \int_y^Y bdy = b(Y-y),$$

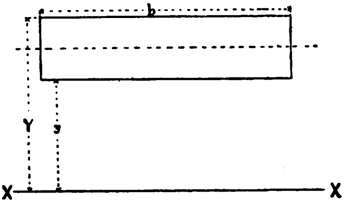

FIG. 20.

and the geometrical moment

$$G = \int_y^Y bydy = \frac{b}{2}(Y^2 - y^2) = b(Y-y)\left(\frac{Y+y}{2}\right).$$

Now $b(Y - y) = A$, and $\frac{Y+y}{2}$ is the distance of the centre of gravity from the given axis. Call it $\bar{x}$; then, $A\bar{x} = G$. The area of the rectangle into the distance of the centre of gravity from the given axis equals the geometrical moment. This is true for any section, and can generally be expressed thus:

$$\bar{x} = \frac{\int bydy}{\int bdy}.$$

For the moment of inertia

$$I = \int_y^Y by^2dy = \frac{b}{3}(Y^3 - y^3).$$

Now any section can be divided into a number of rectangles; and by calculating the value of the area, geometrical moment, and the moment of inertia of each rectangle separately, and adding the results together, the above quantities are determined for the whole section.

An illustration of the method of calculation for a particular section is given in the table below.

b	Y	$Y-y$	*Area.*	Y^2	Y^2-y^2	$\frac{1}{2}b(Y^2-y^2)$	Y^3	Y^3-y^3	$\frac{1}{3}b(Y^3-y^3)$
	0			0			0		
12		3	36		9	54		27	108
	3			9			27		
6		3	18		27	81		189	378
	6			36			216		
1·5		10	15		220	165		3880	1940
	16			256			4096		
4		2	8		68	136		1736	2315
	18			324			5832		
8		2	16		76	304		2168	5781
	20			400			8000		

$A = 93$ $G_A = 740$ $I_A = 10522$

The dimensions are marked on fig 21. Any convenient axis is taken in this case *AA*, from which the distances are measured to the sides of the rectangles. The table is now made on which the values of the areas, geometrical moments, and moments of inertia of the different rectangles are entered, the summation giving the values round the axis *AA* for the whole section.

To determine the neutral axis or the centre of gravity of the section

$$y_a = \bar{x} = \frac{G}{A} = \frac{740}{93} = 8'', \text{ and } y_b = 20 - 8 = 12.$$

For the moment of inertia the equation

$$I_A = I + Ay^2 \quad \text{or} \quad I = I_A - Ay^2 = 10522 - 93 \times \left(\frac{740}{93}\right)^2 = 4634.$$

The moment of inertia of the section, and the working strength of the material, being known, the moment of resistance can be determined. From § 31 the equation for the moment of resistance is $M = \frac{pI}{y}$. Now in the example given there are two values of y, the distances of the top and bottom layers y_a and y_b, respectively, and the stresses on these layers will be different. Now

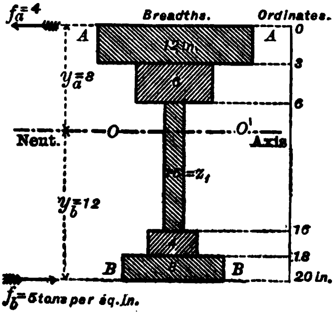

FIG. 21.

$$p_a = \frac{My_a}{I}, \quad \text{and} \quad p_b = \frac{My_b}{I};$$

that is,

$$\frac{p_a}{p_b} = \frac{y_a}{y_b}, \quad \text{and} \quad \frac{y_a}{y_b} = \frac{2}{3};$$

therefore

$$\frac{p_a}{p_b} = \frac{2}{3}.$$

Suppose the working strengths of the material to be $f_a = 4$ tons per square inch in compression, and $f_b = 5$ tons in tension. Assume the upper layers to be in compression, and the lower in tension; then

$$M = \frac{f_a I}{y_a} = \frac{4}{8} \times 4654 = 2317 \text{ in. tons},$$

$$M_1 = \frac{f_b I}{y_b} = \frac{5}{12} \times 4654 = 1931 \text{ in. tons}.$$

The lesser of the two values is to be taken, that is, the lower layer comes to its working stress f_b; but the top layer does not come to its working stress f_a. If $\frac{y_a}{y_b}$ had been in the ratio of $\frac{4}{5}$, then the upper and lower layers would have come to their working stress simultaneously.

The use of unsymmetrical sections is necessary when dealing with materials like cast iron, whose strength to compression differs considerably from that to tension. For mild steel and wrought iron, the strengths to tension and compression are sensibly the same, and consequently the sections are, as a rule, symmetrical.

33. **Radius of Gyration.**—The moment of inertia admits of being expressed in terms of the area of the cross-section and a quantity called the radius of gyration; the relation is expressed by the equation

$$A\rho^2 = I, \quad \text{or} \quad \rho = \sqrt{\frac{I}{A}},$$

where ρ is the radius of gyration, A the total area of the section, and I the moment of inertia.

For sections of girders such as shown in fig. 22, where the material is concentrated in the flanges, the value of ρ approaches $\frac{h}{2}$. Neglecting the web, the moment of resistance can approximately be represented thus:

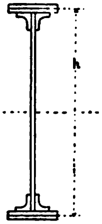

FIG. 22.

$$M = \frac{pI}{y} = \frac{pA\rho^2}{y}, \text{ if } y = \rho \text{ nearly,}$$

$$= pAy\,; \text{ but } y = \frac{h}{2};$$

and A is the total area of the section $= 2a$, where $a =$ area of cross-section of one flange; also $p = f$, the working strength of the material; and substituting

$$M = fah,$$

this formula is useful for the design of the booms of girders, and, with care, the approximation gives sufficiently accurate results.

34. **Deflection of Beams.**—The amount by which any point on the neutral axis in a beam moves under the action of the load, in a direction perpendicular to the span, is called the

deflection at that point. In addition to a beam being sufficiently strong to carry a given load, it is necessary that the deflection should not exceed certain limits; in other words, that the beam should be sufficiently stiff. The measure of the stiffness of a beam is the ratio of the $\frac{\text{deflection}}{\text{span}}$.

Let a beam (fig. 23) be of uniform section and bent by means of couples applied at the ends. It will bend into an arc of a circle, if R is the radius of the circle after bending, l the length of the span or chord, and $\delta = NK$ the deflection; then $\delta\ (2R - \delta) = \frac{l^2}{4}$, because the chords in the circle intersect, and the rectangles under the segments are equal. Now δ is small compared with R: hence, its square can be neglected, and the equation takes the form $\delta = \frac{l^2}{8R}$. From the equation $\frac{p}{y} = \frac{M}{I} = \frac{E}{R}$ of § 31, the value of $\frac{1}{R} = \frac{M}{EI}$, substituting in the formula for the deflection $\delta = \frac{Ml^2}{8EI}$.

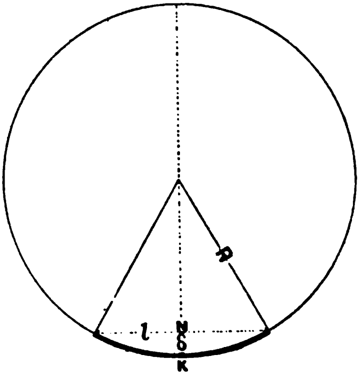

FIG. 23.

The formula just obtained also applies to a girder of uniform strength; that is, a girder of uniform depth in which, for any section, the ratio $\frac{M}{I}$ = constant: the form of the deflection curve in which the girder bends is then the arc of a circle. Now, the majority of girders in practice are designed to carry uniform live and dead loads distributed over the span, and are nearly girders of uniform strength for such loads; and the preceding formula applies, if U and W are the distributed live and dead loads over the span; then, $M = \frac{(U + W)\,l}{8}$ at the centre of the span. Substituting, the formula then takes the form $\delta = \frac{(W + U)\,l^3}{64EI}$, and applies to the majority of plate-girders resting on two supports.

35. **Determination of the Curve of Deflection.**—For a girder or beam other than in the two cases just given, the determination of the curve of deflection requires the use of the calculus. The expression for the radius of curvature of a curve is

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

The solution of this equation applies to few cases; and, to put it in a workable form, the value of $\left(\frac{dy}{dx}\right)^2$ in the numerator is neglected. The tangent of the slope of the curve $= \frac{dy}{dx}$, and is small in all cases of beams and girders, owing to the smallness of the deflection; for this reason its square can be discarded in comparison with unity; the equation then takes the form

$$\frac{1}{R} = \frac{d^2y}{dx^2}; \text{ but } \frac{1}{R} = \frac{M}{EI}.$$

Substitute this value, and the fundamental equation

$$\frac{d^2y}{dx^2} = \frac{M}{EI},$$

for all cases of transverse loading, is obtained.

To apply the equation, express M as a function of x; then, by successive integration, the slope and form of the deflection curve is obtained. Thus, let a beam of uniform section be loaded with a weight in the middle (fig. 24), and take the centre D as origin, and the horizontal as the axis of X; then, if l is the length of the beam,

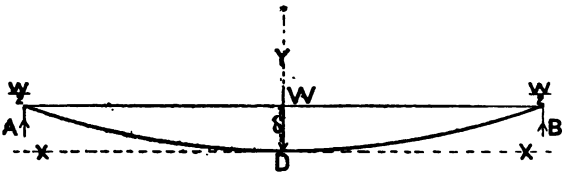

FIG. 24.

$$EI\frac{d^2y}{dx^2} = M = \frac{W}{2}\left(\frac{l}{2} - x\right).$$

Integrating

$$EI\frac{dy}{dx} = \frac{W}{2}\left(\frac{l}{2}x - \frac{1}{2}x^2\right) + c \text{ when } x = 0,\ \frac{dy}{dx} = 0;\text{ therefore } c = 0.$$

This gives the slope of the curve at any point. Integrating a second time,

$$EIy = \frac{W}{2}\left(\tfrac{1}{4}lx^2 - \tfrac{1}{6}x^3\right) + c \text{ when } x = 0,\ y = 0;\text{ therefore } c = 0.$$

To get the maximum deflection, put $x = \frac{l}{2}$; then

$$\delta = \frac{\frac{W}{2}\left(\frac{1}{16}l^3 - \frac{1}{48}l^3\right)}{EI} = \frac{1}{48}\frac{Wl^3}{EI}.$$

The maximum deflection of a beam can generally be expressed in the form $\delta = n\frac{Wl^3}{EI}$, n being a coefficient determined from the analysis, the method of proceeding being similar in all cases to that given above.

For a beam of uniform section uniformly loaded $n = \frac{5}{384}$. These two cases are useful for determining the deflection of rolled beams and troughs.

If the deflection of a beam is determined for a particular loading, and another distribution of loading is superimposed upon it, then the total deflection at any point will be the sum of the deflections due to the separate loads. A number of cases fully worked out will be found in Alexander and Thomson's "Applied Mechanics," Dubois' "Graphic Statics," and other well-known treatises.

35A. **Proof Deflection.**—The proof deflection of a beam is the deflection when the stress on the outer layers of the beam comes to the maximum safe-working strength of the material. From the general formula for the deflection of § 35, and neglecting the coefficient n, the deflection $\delta_0 \propto \frac{Wl^3}{EI}$. In this expression the factor Wl varies directly as the bending moment M for any *similar* distribution of loading: hence, under these conditions, $\delta_0 \propto \frac{Ml^2}{EI}$, the expression being of the same form as that given for the deflection of a uniform beam bent by a couple

(§ 34). If now M and I are constant, the deflection varies directly as the square of the length.

From the general relation of § 31, $\frac{M}{I} = \frac{p}{y}$. Suppose now in this equation that p is kept constant, and also that the value of y depends on I, that is, the sections of the beams to be compared are kept *similar*; then, if y varies, M varies with I to preserve the equality of the ratio. Under these conditions $\frac{p}{y}$ can be substituted for $\frac{M}{I}$ in the above relation; hence $\delta_0 \propto \frac{pl^2}{Ey}$. Now, p and E are constants; therefore $\delta_0 \propto \frac{l^2}{y}$; or, in words, the proof deflection varies directly as the square of the length, and inversely as the depth.

36. **Distribution of Stress.**—On fig. 17 a bar was shown with a force P acting on it in the direction of its length; the force was assumed to be equally distributed over the area of the

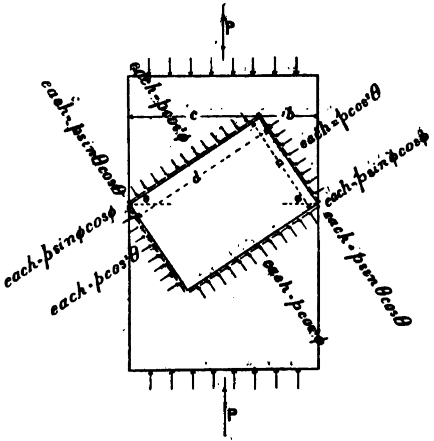

FIG. 25.

cross-section A; and the intensity p of the stress per unit of area was $p = \frac{P}{A}$. Under the action of this force the material of the bar is in a state of internal stress. It is required to determine the intensity of these stresses on any plane within the bar. Consider the bar shown on fig. 25, and let it be of unit thickness, and rectangular in section. Draw in any position on the face of the bar a rectangle; and let it be the base of a rectangular prism which pierces it in a direction perpendicular to the plane of the paper. Consider the stresses acting on the face of this prism. On the face a there is a stress pb; resolve it into two directions perpendicular and tangential to the face a; then, if $\theta =$ the angle between the axis of the bar and the perpendicular to the face a, the normal stress $= pb \cos\theta$, and the intensity of the normal stress per unit of area $p_n = \dfrac{pb \cos\theta}{a} = p \cos^2\theta$. The tangential stress $= pb \sin\theta$, and its intensity per unit area

$$p_t = \frac{pb \sin\theta}{a} = p \sin\theta \cos\theta.$$

Similarly on the face d the intensity of the normal stress $p'_n = p \cos^2\phi$, and of the tangential stress $p'_t = p \sin\phi \cos\phi$. Now the sum of $\theta + \phi =$ a right angle; therefore, $\sin\theta = \cos\phi$, and $\sin\phi = \cos\theta$, and $p \sin\theta \cos\theta = p \sin\phi \cos\phi$; that is, the tangential stresses per unit area on any two faces at right angles within the solid are of equal intensity. This result can also be shown thus: the tangential stresses acting on the face a and the face opposite to it, form a couple whose moment $= p_1ad$, and the tangential stresses on the face d and the face opposite to it, form a couple tending to turn in the opposite direction of p_2da, where p_1 and p_2 are the intensity of the tangential stresses on the faces a and d, respectively. Since the solid is in equilibrium, these couples must be equal, that is, $p_1ad = p_2da$, or $p_1 = p_2$. The sum of the normal stresses $= p \cos^2\theta + p \sin^2\theta = p$; that is, the sum of the normal stresses on any two planes at right angles within the solid is equal to the applied stress.

37. **Compound Stress.**—On fig. 26 a rectangular block of unit thickness is shown, on which forces of intensity p, a thrust, are shown, acting on a pair of parallel faces, and forces of intensity q, a pull, act on another pair of parallel faces. (By mistake, arrows q show a thrust on the figure.) Consider the stresses acting on the faces of a rectangular prism, as in § 36, and resolving the forces perpendicular to and parallel to the faces; then the normal stress on the face $a = p_n = p \cos^2\theta - q \sin^2\theta$, and on the face $d = p'_n = p \cos^2\phi - q \sin^2\phi$. The tangential stresses on the face $a = p_t = (p + q) \sin\theta \cos\theta$, and on $d = p'_t = (p + q) \sin\phi \cos\phi$. As before, $\theta + \phi = 90$; and therefore the tangential stresses are equal to one another, or

$p_t = p'_t$. To determine the intensity and the obliquity of the stress which would replace these normal and tangential stresses, on fig. 27 a small portion of the face a of fig. 26 is shown, together with the normal and tangential stresses acting thereon,

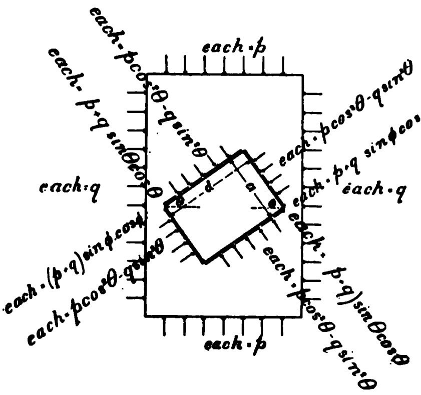

FIG. 26.

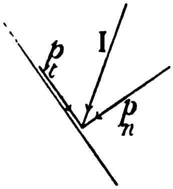

FIG. 27.

the resultant intensity I will be obtained by compounding these stresses. Since the normal and tangential stresses are at right angles,

$$I = \sqrt{(p\cos^2\theta - q\sin^2\theta)^2 + (p+q)^2\sin^2\theta\cos^2\theta}$$

$$= \sqrt{p^2\cos^2\theta + q^2\sin^2\theta},$$

and the tangent of the angle of obliquity—that is, the tangent of the angle between the normal and the direction of the stress—equals

$$\tan\gamma = \frac{(p+q)\sin\theta\cos\theta}{p\cos^2\theta - q\sin^2\theta}.$$

If now a number of values of θ are taken, and the values of I obtained and plotted to scale, the extremities of I will be found to lie on an ellipse, of which p and q are the semi-major and semi-minor axes. The ellipse so determined is called by Rankine the ellipse of stress, because its semi-diameter in any direction represents the intensity of the stress in that direction. It does not matter what the ratios of p to q are, or whether

p and q are of the same or opposite signs—that is, both tensions or thrusts, or one a tension and the other a thrust—the result will always be an ellipse. If $q = 0$, the ellipse becomes a straight line, and when $p = q$, a circle.

37A. **Pure Shear.**—If the intensity of the thrust p equals the intensity of the pull q, that is $p = -q$, and the angle $\theta = 45°$, then the stress on that plane is one of pure shear. From the equation for the normal intensity, $p_n = p'_n = 0$, and from the equation for the tangential intensity, p_t is a maximum; because $\sin\theta \cos\theta$ is a maximum and equal to $\frac{1}{2}$. Substituting $-q = p$, the equation for the tangential intensity is $p_t = 2p \times \frac{1}{2} = p$; that is, the shear stress is of the same intensity as the applied stresses. In loaded beams the stress at the neutral axis is of this kind. Also when a shaft is subjected to torsion in planes perpendicular to the axis, the stress is one of pure shear.

38. **Principal Stresses—Axes of Stress.**—It is now proposed to show that any state of stress, in two dimensions, may always be replaced by a pair of stresses acting at right angles to one another. Let any two stresses be given in any directions and of any magnitude. Draw any two planes AB, BC, at right angles to one another (fig. 28) on which the stresses act, and resolve these stresses, normal and tangential, to the given planes, as in § 37.

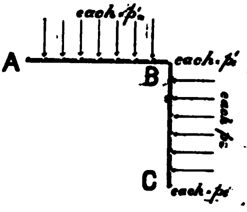

Fig. 28.

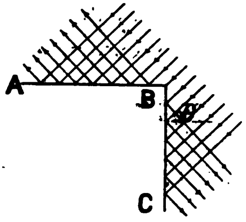

Fig. 29.

Let the sum of the intensities of the normal components be p'_n on the plane AB, and p_n on the plane BC. Also let the sum of the intensities of the tangential components be p'_t on the plane AB, and p_t on the plane BC. Since these planes are at right angles to one another, $p'_t = p_t$.

Now, if it is possible that two stresses, at right angles to one another, can replace the given stresses, they must fulfil the conditions obtained in § 37; that is,

$$p_n = p\cos^2\theta \pm q\sin^2\theta, \quad p'_n = p\sin^2\theta \pm q\cos^2\theta,$$

and $$p_t = p'_t = (p \mp q)\sin\theta\cos\theta;$$

the upper and lower signs being taken according as the stresses are of the same or different kinds. To determine the angle θ from these equations, subtract the first two equations from one another, and divide the result into the third, thus:

$$\frac{p_t}{p_n - p'_n} = \frac{(p \mp q)\sin\theta\cos\theta}{(p \mp q)(\cos^2\theta - \sin^2\theta)} = \frac{\left(\frac{p \mp q}{2}\right)\sin 2\theta}{(p \mp q)\cos 2\theta};$$

therefore $$\frac{2p_t}{p_n - p'_n} = \tan 2\theta. \qquad (1)$$

This equation always gives two values of θ differing by 90°, showing that two planes at right angles can be found on which the stresses p and q are wholly normal.

To determine the magnitude of the stresses, re-write the equations for p_n and p'_n in the form

$$p - p_n = (p \mp q)\sin^2\theta \quad \text{and} \quad p - p'_n = (p \mp q)\cos^2\theta,$$

and multiply the equations together thus:

$$(p - p_n)(p - p'_n) = (p \pm q)^2\sin^2\theta\cos^2\theta = p_t^2. \qquad (2)$$

A similar equation can be deduced for q. The roots of this quadratic equation give the required stresses p and q. Having now determined the quantities θ, p, and q, the ellipse of stress can be drawn: p and q are called Principal Stresses, and their directions the Axes of Stress.

Fig. 29 shows the direction and kind of stresses which would replace the stresses given on fig. 28. The particular case in which one of the normal stresses is zero occurs when a direct stress and a shearing stress are given. Let $p'_n = 0$, and put

$s = p_t$ the shearing stress; equations 1 and 2 may then be written

$$p_n \tan 2\theta = 2s, \qquad (3)$$

$$p(p - p_n) = s^2;$$

therefore

$$p = \frac{p_n}{2} \pm \sqrt{\frac{p_n^2}{4} + s^2}. \qquad (4)$$

Of the roots of the quadratic, the greater stress has the same sign as that of p_n, and the other the opposite. By eliminating p_n from the above equations the value of $\tan\theta = \frac{s}{p}$ is obtained. Hence, of the two values of θ given by equation 3, the one less than 45° must correspond to the greater value of p, and the major principal stress is of the same kind as p_n, and inclined to it at an angle less than 45°.

39. **Approximate Distribution of Bending and Shear Stress in I-section Beams and Girder.**—The majority of beams and plate-girders in practice are of this section, and when loaded are in a state of compound stress, the material having a double function to perform in resisting the bending moments and shear stresses. On fig. 30, one-half of a beam of I-section with equal flanges is shown loaded at the centre with a weight W. It is sufficient, as a first approximation, to assume that the bending moments are taken up by the flanges, and that the shear stress is uniformly distributed over the web. Take K_1 and K_2, two transverse sections of the web, at distances X_1 and X_2 from the centre of the beam, and let $2a$ = the span; then the bending moment at K_1 is

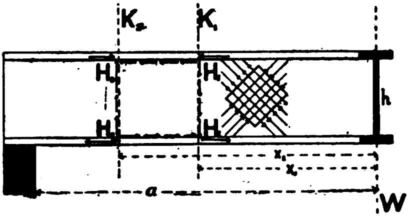

FIG. 30.

$$M_1 = \tfrac{1}{2}W(a - x_1), \quad \text{and at } K_2 \quad M_2 = \tfrac{1}{2}W(a - x_2).$$

Now, the flanges are assumed to take up the whole of the bending stress: hence, if H is the stress on one of the flanges, $Hh = M$, where h is the depth of the girder (§ 33); therefore at K_1 the value of

$$H_1 = \frac{W(a - x_1)}{2h}, \quad \text{and at} \quad H_2 = \frac{W(a - x_2)}{2h},$$

similar values of H_1 and H_2, but in the reverse direction, acting on the lower flange. The difference of these two stresses

$$H_1 - H_2 = \frac{W}{2h}(x_2 - x_1)$$

tends to push the upper edge of the web in one direction, and the lower edge in the opposite direction; on the other hand, the shear stress on the web at the sections K_1 and K_2 tends to turn the web in the opposite direction, and these couples acting along the edges of the rectangle must be in equilibrium. Now, it has already been shown that, on any two planes at right angles, the tangential stresses are of equal intensity (§ 36): hence the tangential stresses on the upper and lower edges of the web joining the flange must be the same as the shear stress on the edge of the web at right angles to the flanges. This can also be shown thus: the moment

$$(H_1 - H_2)\,h = \frac{W}{2}(x_2 - x_1);$$

let s' be the intensity of shear stress on the upper and lower edges of the web, and t the thickness; then

$$s't\,(x_2 - x_1) = H_1 - H_2:$$

hence the moment

$$(H_1 - H_2)\,h = s't\,(x_2 - x_1)\,h.$$

Let the intensity of the shear stress on the web $= s$; then $sth = \frac{W}{2} =$ the shear stress at the section, and its moment $= sth\,(x_2 - x_1)$; and these moments are equal for equilibrium, or

$$s_1th\,(x_2 - x_1) = sth\,(x_2 - x_1):$$

therefore, $s_1 = s$.

It follows from this, since the normal reactions on the rectangle are zero, that the web is subjected to a pure shear stress.

Now, a pure shear stress is made up of two direct stresses of equal intensity, and of opposite signs acting at right angles to one another (§ 37); hence the web, under these conditions, is subjected to thrust and tension on lines inclined at 45° to the axis, somewhat as shown on fig. 30. Since the flanges are supposed to take up the bending, the area of any cross-section is proportional to the bending moment at that cross-section, and the area of the web at any cross-section is proportional to the shear stress at that section.

40. **Distribution of Shear Stress.**—In beams and plate-girders the bending moments and shear stresses give rise to a state of compound stress throughout the material; the stresses due to the bending moments have already been investigated, and the general equation $\frac{p}{y} = \frac{M}{I}$ was obtained in § 31: the distribution of the shear stress now requires investigation. On fig. 31 a portion of a rectangular beam is shown which is under stress, due to some distribution of loading; two sections, AA', BB', are taken at a distance of δx apart; at AA' there is a bending moment M_1, and at BB' a bending moment M_2, M_1 being $> M_2$.

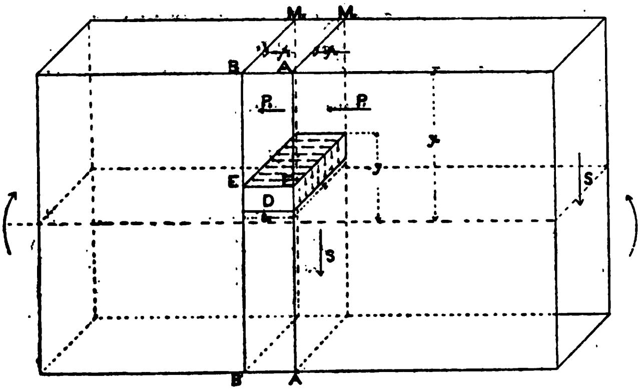

Fig. 31.

Consider the section AA': it is exposed to a shear stress distributed over it, the total amount of the shear stress being represented by the arrow S. Part of this shear stress is taken up by the face of the parallelepiped O in the plane A. Now, whatever may be the intensity of the shear stress on this face, it must be of the same intensity as the shear stress acting on the upper face EE of the block (§ 37); hence, if the shear stress on this face is known, it is determined for the other face also.

Let P_1 and P_2 be the total stresses on the faces AE, BE, respectively; their difference $P_1 - P_2$ produces a shear stress on the face EE of the square block. Let s be the intensity of that shear stress, and $z\delta x$ the area of the face EE, z being the breadth of the beam: then $sz\delta x = P_1 - P_2$.

To determine P_1 consider the section AA_1. The stress p on any layer is proportional to its distance y from the neutral axis; hence

$$\frac{p}{y} = \frac{p_a}{y_a}, \quad \text{or} \quad p = \frac{p_a}{y_a} y.$$

Now, the stress over the face AE is

$$P_1 = \int_y^{y_a} pzdy = \frac{p_a}{y_a} \int_y^{y_a} zydy = \frac{p_a}{y_a} G,$$

where G is the geometrical moment (§ 30) of the face AE with regard to the neutral axis. Similarly

$$P_2 = \frac{p_b}{y_a} \int_y^{y_a} zydy = \frac{p_b}{y_a} G,$$

p_b being the stress on the outer layer at the section BB'. Substituting for P_1 and P_2,

$$sz\delta x = P_1 - P_2 = \frac{p_a - p_b}{y_a} G;$$

but
$$\frac{p_a}{y_a} = \frac{M_1}{I}, \quad \text{and} \quad \frac{p_b}{y_a} = \frac{M_2}{I};$$

therefore
$$\frac{p_a - p_b}{y_a} = \frac{M_1 - M_2}{I} = \frac{\Delta M}{I},$$

where ΔM is the small difference of the bending moments on the sections AA', BB', at a distance δx apart. Substituting, the equation obtained is

$$sz\delta x = \frac{\Delta M}{I} G, \quad \text{or} \quad s = \frac{\left(\frac{\Delta M}{\delta x}\right) G}{Iz};$$

but in the limit $\frac{\Delta M}{\delta x} = F$, the shearing force at the section: hence,

$$s = \frac{FG}{Iz},$$

where I is the moment of inertia of the section about the neutral axis, and z the breadth of the section at the parallelepiped. The value of the intensity of the shearing stress s, so determined for the upper face EE of the block, is also the shearing stress on the face of the block in the section AA'. Also, since the parallelepiped is supposed to be indefinitely small, the shearing stresses on its other two faces are of the same intensity, but of opposite sign. If the shearing force be evaluated for any position of the parallelepiped in the face AA', the value of s will be proportional to the geometrical moment G, and inversely proportional to the breadth z of the cross-section, F and I being constant for any one cross-section. Now, in similar sections, the ratios of G and z to one another are the same: hence, the distribution of the shearing stress s depends only on the form of the cross-section.

41. **Rectangular Cross-Section.**—Let the rectangle shown on fig. 32 be the cross-section of a beam. Using the result of the preceding investigation, what is the distribution of the shearing stress? Note in what follows that $s = q$, $s' = q'$, and $s_0 = q_0$ on figs. 32 and 33.

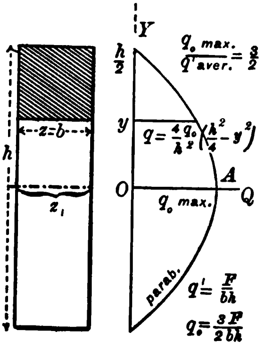

FIG. 32.

Since $z = b =$ constant, s will vary directly as the geometrical moment G. Now,

$$G = \int_y^{\frac{h}{2}} by\,dy = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

(§ 30). This is a maximum when $y = 0$; that is, for one-half the section round the neutral axis. When $y = \pm\frac{h}{2}$, the geometrical moment $= 0$; that is, when the cross-sections become equal to nothing at the top and bottom of the beam. Putting in the above value of G in the equation, and for I its value $\frac{1}{12}bh^3$,

$$s = \frac{F}{I}\,\frac{G}{z} = \frac{F\frac{b}{2}\left(\frac{h^2}{4} - y^2\right)}{\frac{1}{12}bh^3 \,.\, b}$$

when $y = 0$; this reduces to

$$s_0 = \frac{3}{2}\frac{F}{bh}, \quad \text{or} \quad F = \frac{2bhs_0}{3}.$$

Substitute this value for F in the above equation; then

$$s = \frac{4s_0}{h^2}\left(\frac{h^2}{4} - y^2\right).$$

Suppose the shearing stress to be equally distributed over the cross-section, then its average intensity

$$s' = \frac{F}{bh}, \quad \text{and the maximum intensity is} \quad s_0 = \frac{3}{2}\frac{F}{bh}$$

at the neutral axis: hence, the average is to the maximum intensity in the proportion of 2 : 3.

On drawing the curve showing the distribution of the shearing stress at any cross-section, the parabola shown on the left-hand side of fig. 32 is obtained. The ordinates OY, OA are drawn at right angles, OA being in line with the neutral axis. Marking off $OA = s_0$ to some scale, and plotting other values of s perpendicular to OY to the same scale, the curve showing the distribution of the shearing stress can at once be determined.

42. **Hollow Rectangular Section or Double T-Section.**—The sections shown on fig. 33 are important, as they approach the form of the cross-sections of beams and

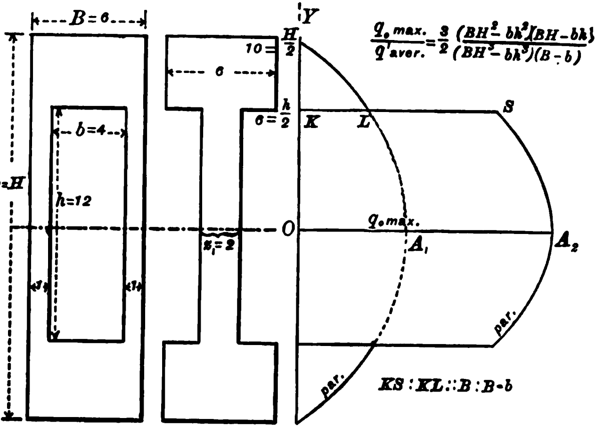
B = 6
20 = H
b = 4
h = 12
1
1
10 = H/2
6
6 = h/2
$s_1 = 2$
Y
O
K
L
S
q_0 max.
A_1
A_2
par.
par.
$\frac{q_0 \text{ max.}}{q' \text{ aver.}} = \frac{3}{2}\frac{(BH^2 - bh^2)(BH - bh)}{(BH^3 - bh^3)(B - b)}$
KS : KL :: B : B - b

FIG. 33.

plate-girders in practice. To determine the distribution of the shearing stress, using the lettering and dimensions shown on the figures, the following are the required data:—

Area of cross-section $= A = BH - bh$.

$G = \frac{B}{2}\left(\frac{H^2}{4} - y^2\right)$ for values of y between $\pm\left(\frac{H}{2}\right.$ and $\left.\frac{h}{2}\right)$, and

$G = \frac{B}{8}(H^2 - h^2) + \left(\frac{B-b}{2}\right)\left(\frac{h^2}{4} - y^2\right)$ for values of y between $\pm\frac{h}{2}$ and 0.

$z = B$ for values of y between $\pm\left(\frac{H}{2}\right.$ and $\left.\frac{h}{2}\right)$, and $(B-b)$ for values between $\pm\frac{h}{2}$ and 0.

$I = \frac{1}{12}(BH^3 - bh^3)$ for the total section.

Putting in the above values, the equation for the shearing stress becomes, for values of y between $\pm\left(\frac{H}{2}\right.$ and $\left.\frac{h}{2}\right)$,

$$s = \frac{F}{I}\,\frac{G}{z} = \frac{F \cdot \frac{B}{2}\left(\frac{H^2}{4} - y^2\right)}{\frac{1}{12}(BH^3 - bh^3)(B)};$$

and for values of y between $\pm\frac{h}{2}$ and 0,

$$s = \frac{F}{I}\,\frac{G}{z} = \frac{F\left[\frac{B}{8}(H^2 - h^2) + \left(\frac{B-b}{2}\right)\left(\frac{h^2}{4} - y^2\right)\right]}{\frac{1}{12}(BH^3 - bh^3)(B-b)}.$$

The maximum value of the shearing force is, as before, at the neutral axis. Putting $y = 0$, then

$$s_0 = \frac{F}{I}\,\frac{G_0}{z} = \frac{3}{2}\cdot\frac{F(BH^2 - bh^2)}{(BH^3 - bh^3)(B - b)},$$

and the average intensity is

$$s' = \frac{F}{BH - bh};$$

therefore,
$$\frac{s_0}{s'} = \frac{3}{2}\frac{(BH^2 - bh^2)(BH - bh)}{(BH^3 - bh^3)(B - b)}.$$

For the dimensions given on fig. 33,

$$s_0 = \cdot 033F, \quad \frac{s_0}{s'} = 2\cdot 4.$$

The locus shown on fig. 33 gives the distribution of the shearing stress on the cross-section; it also gives the distribution of the shearing stress on the horizontal layers of the beam at that cross-section. Now, in passing from the horizontal layer at *K* from the flange to the web, there is a sudden change in intensity from *KL* to *KS*: such a sudden change cannot and does not take place in the actual beam; and hence the investigation only points out approximately the real distribution of the shearing stress on such irregular sections. Any sudden change in the intensity of the shearing force would require the horizontal shearing force at *K* to be infinite in amount; whereas, in reality, the change takes place continuously in passing from layer to layer. The investigation, however, shows that any sudden change in the shape of the section always involves a great change in the shearing stresses; and for that reason corners should be always filled in, and sharp changes of section as far as possible avoided. If the corners of *S* and *L* are rounded off so as to show a continuous change of the shearing force, it will give some idea of its true distribution.

43. **Lines of Stress in Beams and Plate-Girders.**—It was shown in § 38 that any state of compound stress in two dimensions could always be reduced to two simple stresses acting at right angles to one another; also, the particular case in which one normal stress together with a shear stress was examined. Now, consider the equilibrium of the parallelepiped D of fig. 31, and shown repeated on fig. 34: it is exposed to two different kinds of stress—a shear stress of known intensity, § 40, and to a direct stress p_n, also of known

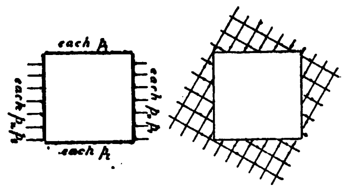

FIG. 34.

intensity, given from the relation $\frac{p}{y} = \frac{M}{I}$ of § 31; if, then, these two stresses be evaluated at any point, the directions and intensities of the two principal stresses can be determined. Thus, if s is the intensity of the shear stress, and p_n the intensity of the normal stress, the principal stresses are given by the equation

$$p = \frac{p_n}{2} \pm \sqrt{\frac{p_n^2}{4} + s^2} \quad \text{of § 38,}$$

the greater principal stress being of the same sign as p_n; the angle of inclination θ is given by the equations

$$\tan 2\theta = \frac{2s}{p_n}, \quad \text{and} \quad \tan \theta = \frac{s}{p},$$

attention being paid to the signs of the stresses. On drawing the directions of the principal stresses thus evaluated at any point, and then again evaluating the principal stresses at a point near to that point, and taken on the line of direction of one of the principal stresses, and so on for a number of points, a curve will be obtained which is called 'a line of Principal Stress.' On fig. 35, these lines of principal stress are shown for a rectangular beam loaded at the centre, and on fig. 36, for a plate-girder, loaded at the centre—a little more than one-half the girder being shown. The curves have the following properties: they cut the neutral axis at 45°, the bending moment there being zero, $p_n = 0$; and the stress is one of pure shear, $2\theta = 90°$, and $\theta = 45°$. They intersect at 90°, the tangents giving the directions of the principal stresses at the point. They meet the upper and lower edges of the beam at 90° and 0°, respectively; on fig. 36 the lines of thrust are shown by thick lines,

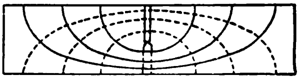

FIG. 35.

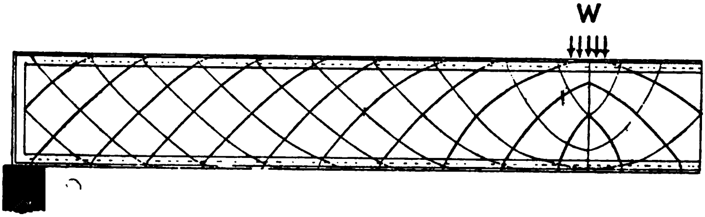

FIG. 36.

and the lines of tension by thin lines crossing the lines of thrust at right angles. If figs. 35 and 36 are compared together, it will be noticed that the lines of stress of the plate-girder are inclined approximately at 45° throughout the web, except for the small portion of the girder near the centre, and that these lines of stress rapidly change their inclination in passing from the web to the flanges, § 42; whereas, in the rectangular section, there is no sudden change of curvature in the lines of stress, the change being continuous, and the inclination of the curves only approximating to 45° near the neutral axis.

44. **Practical Deductions.**—The foregoing investigation leads to the conclusion that, in the design of plate-girders, it is sufficient to assume, as in § 39, that the flanges resist the bending moments, and the web resists the shear, and that it is uniformly distributed over the cross-section, the lines of stress being inclined at 45°. These theoretical deductions were strikingly confirmed in the experiments carried out on the large model of the Britannia Tubular Bridge,* the lines of stress being shown by experiment to slope approximately at 45°, the sides of the girder being drawn into waves along these lines.

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* Clark, on the Britannia and Conway Tubular Bridge, pp. 181–3.

THE DESIGN OF PLATE-GIRDERS.

45. **Working Strength to Shear, and Distribution of Shear Stress.**—In §§ 39 and 44, it was shown that it is sufficient to assume that the shear stress on any vertical section is uniformly distributed. Consider the plate-girder shown on fig. 38, if any vertical section is taken, and the total shearing force F at that section is known, then the sectional area A of the web will be

$$A = \frac{F}{s}, \quad \text{or} \quad F = As,$$

where s is the intensity of the shear stress. Under these conditions, the web-plate is subjected to equal intensities of thrust and tension, § 37, on lines inclined at 45°.

It was pointed out in § 4 that the strengths of iron and steel to tension, compression, and pure shear are approximately all equal to one another; and this will also be true of the working strengths. Let f_c, f_t, and f_s represent the working strengths of the material to compression, tension, and pure shear, respectively. Since these are all equal, one common symbol f can be used to represent these strengths. Now, the symbols p, q, and s represent the intensities of the stresses to thrust, tension, and pure shear, respectively; and these are all equal. Put the working strengths for the stresses; that is,

$$p = f_c = q = f_t = s = f_s = f;$$

then the formula above for $A = \frac{F}{s}$, substituting f for s, gives $A = \frac{F}{f}$; also the intensities of the stresses to thrust and tension on any two strips such as BB, CC are equal to f; the area of the cross-section A varying with the total shearing force F. Let D = the depth of the girder, and t = the thickness of the web; then

$$A = Dt = \frac{F}{f}.$$

From this it follows that, as D increases, t decreases. This result would be literally true within a wide range of the depth, were it not that the failure of the web due to buckling requires to be considered. It is the addition of the material to the web for this purpose that fixes the economic depth of the girder.

A question of some difficulty requires to be considered here. The strip *CC* (fig. 38) may be looked upon as being placed side by side with the strip *BB*; then, instead of the stresses acting through the solid web as a pure shear stress, the stresses would be transmitted by the strips independently as in a lattice-girder. Would the material, if it were thus disposed, be weaker than in the solid web?

Now, f is the intensity of the stress on the strips *CC*, *BB*, one being in tension and the other in compression.

Neglect, as before, all consideration of the additional material required for stiffening, that is, the material to be added to the strip *BB* to allow it to act as a strut; then the intensity of the shear stress on a vertical section would be

$$f \sin\theta \cos\theta = \frac{f}{2},$$

since $\theta = 45^\circ$ (§ 36). The thickness of the two strips is the same as that of the web: hence this distribution of the material would only transmit one-half of the shear stress, and would require that the thickness should be doubled.

In this work, for the design of the web of plate-girders, the lesser of these two values will be used; that is, the working strength of the web to a shear stress will be $f/2$, and the formula for the area becomes $A = \dfrac{2F}{f}$, this value giving a result more in accordance with practice, and also eliminating much of the uncertainty for the thickness of the web when the relative yielding of the web and struts comes to be considered.

46. Strength of Materials under Compound Stress.—The result contained in § 45 requires that the strengths of the material under a compound stress should be carefully considered. It has been already remarked that the strengths of materials, such as iron and mild steel, to tension, compression, and pure shear, are nearly all equal to one another. These results, obtained from experimental tests, point to the following remarkable deduction, viz.:—That a bar subjected to a direct stress of tension or compression is not weakened by being subjected to stresses of any value less than the given stress either of the same or opposite kind in planes perpendicular to that of the given stress.

A bar of mild steel, subjected to tension or compression, breaks with the same intensity of stress in both cases: consider what happens to the bar when subjected to the torsion test. On planes perpendicular to the length of the specimen, the stress is one of pure shear, that is, it is subjected to direct tensile and compressive stresses; these stresses are equal to one another and to the shear stress; but the breaking shear stress is of the same intensity as that of the bar when broken, either by tension or compression: hence, there are two stresses acting on the material at right angles to one another, either of which would fracture it if applied separately. Experiments on the breaking-strength of spherical shells also point to the same result, the material in this case being subjected to two stresses of the same kind at right angles to one another; and the breaking-strength has been found to be approximately equal to that of the tensile strength of the material. Further experiments are required on materials under compound stress before any generalisation can be made; up to the present, the experiments point to the deduction already given.

47. **The Plate-girder considered as a Lattice-girder.**—In § 45 it was deemed advisable to treat the web of the plate-girder as being made up of separate parts in thrust and tension; from this point of view it may be considered as being derived from the lattice type of girder. Suppose the girder on the left half of fig. 37 to be made up of a web-plate and struts, as shown, and carrying a uniformly-distributed load which is transmitted to the girder at a number of points directly under the struts, placed at equal distances apart, such as W_1, W_2, etc.

Consider the web-plate to be made up of a series of ties transmitting their stresses through the flanges to the struts: such a girder would be of the double triangular or lattice type. Owing to the ties being of considerable breadth, the transmitted tensile stress would give rise to a bending of the flanges round the ends of the struts; the wavy line on fig. 37 shows, in an exaggerated manner, the effect of this bending on the flanges. If any vertical section be taken such as AA, the area of the cross-section of the struts and ties will be, as before,

$$A = \frac{2F}{f},$$

neglecting all consideration of the additional material required

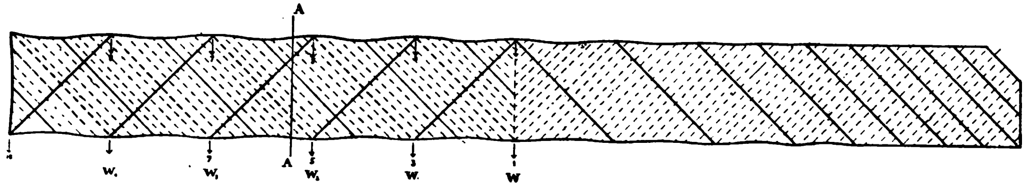

Fig. 37.

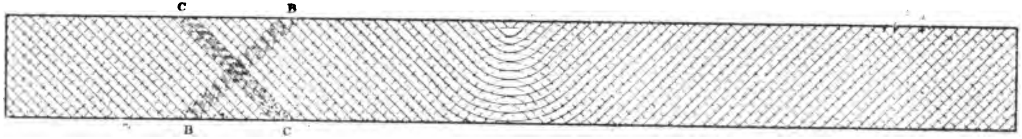

Fig. 38.

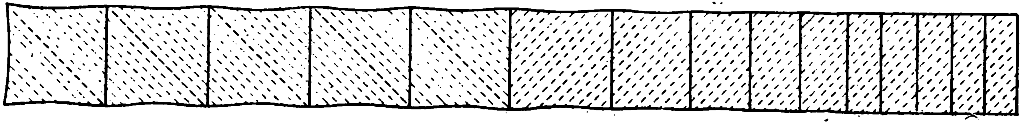

Fig. 39.

for stiffening the struts. This area is divided equally between the web and struts. Now, suppose the material in the struts to be spread equally over the web, then the struts and ties would cross continually, as in fig. 38, the result being a plate-girder. This distribution of the material would not be so efficient, owing to the weakness of the plate, considered as a strut, to transmit thrust, although its cross-section would be of sufficient area to transmit the shearing stress.

On the left half of fig. 39 a plate-girder is shown with vertical struts. As in an N-girder, the web is supposed to be made up of a series of ties transmitting their stresses through the flanges to the struts, the wavy line, as before, showing in an exaggerated way the effect of the bending of the flanges round the struts. Under these circumstances, the cross-section of the web will be, as before,

$$A = \frac{2F}{f};$$

and, in addition to this, there is the material required for the struts.

Suppose these struts to be spread over the web. Resolving the stresses in the web vertically and horizontally along the booms, the additional area, neglecting the material required for stiffening, would be

$$A' = \frac{F}{f}:$$

hence, the volumes of the webs for two similar girders, with vertical and inclined struts, respectively, would be as 2 : 3, the conditions of loading and span being the same in both cases. The material required for stiffening the struts, etc., modifies this ratio considerably, as the analysis given in §§ 65–74 will show.

When considering the plate-girder from this point of view, the web is assumed to be in tension, whereas in reality in this type the web transmits both thrust and tension—the tension being always greater than the thrust; and it is the excess of the tension over the thrust which is transmitted through the flanges to the struts. The same argument, though in a much less degree, applies to the girder of fig. 38, with struts of 45°, the struts in this case taking the greater part of the thrust.

Hitherto, the tendency in the design of plate-girders has been to look upon the web as transmitting the shear stress; and that the object of placing stiffeners on the web was to stiffen it for that purpose. Considered from this point of view, the stiffeners play an important part, as well as struts, and must be designed accordingly.

The exact proportion between the stresses of thrust and tension in the web is a problem of almost hopeless complexity, and can only be approximately solved; however, as a rule, so small a proportion of the thrust may be considered to be taken up by the web, that in most cases it can be neglected with little error in the final design.

48. **The Web considered with Struts.**—On figs. 37, 38, 39, three types of plate-girders were shown. Consider a portion of the web of fig. 37 with struts at 45°, as shown enlarged on fig. 40. Under the applied forces of thrust and

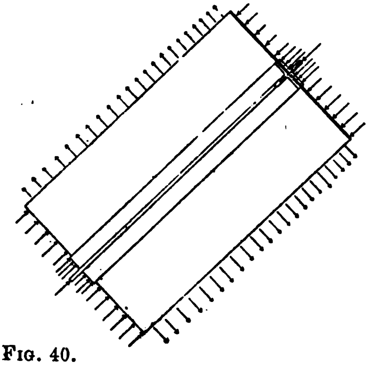

Fig. 40.

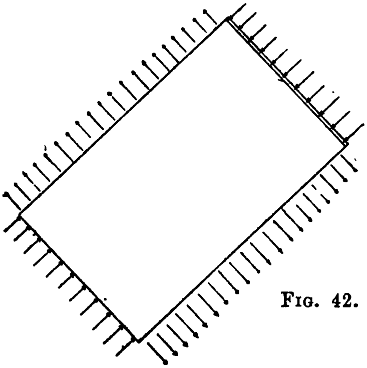

Fig. 42.

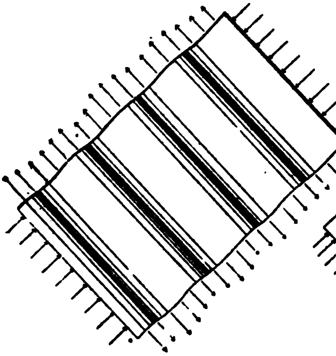

Fig. 41.

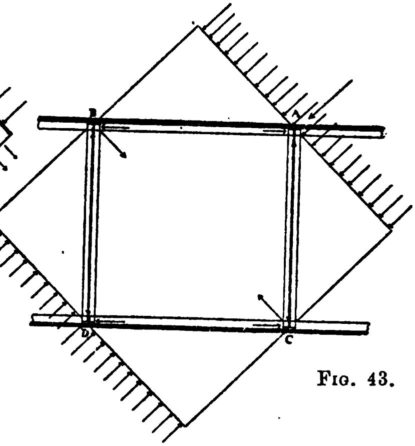

Fig. 43.

tension, the web would yield, throwing the thrust on the struts; that is, the web, being flexible, yields relatively to the strut, and the greater part of the thrust comes on the strut. As already remarked (§ 44), the experiments on the model of the Britannia Bridge showed that, under these circumstances, the web tends to break up into waves somewhat as on fig. 41. If now the material in the struts is spread over the web, as in fig. 42, it is in the same condition as in the girder of fig. 38, and transmits both thrust and tension, and the web itself must be considered as a strut; lastly, the struts are vertical, as on fig. 43. One bay of the girder of fig. 39 being shown, the greater part of the thrust is transferred to the joint *A*, resolution of the stresses taking place along the struts and flanges, as shown at *B*, *C*, *D*, respectively, the web parallel to the line *BC* being in tension and thrown into waves at right angles to it.

49. **Criticism of the Analysis.**—The design of plate-girders upon the lines indicated shows a marked saving of material in the web as compared with girders as at present designed; and for that reason the validity of the assumptions should be criticised.

In taking the working value of $f/2$, instead of f (§ 45), this gives a factor on the side of safety. In most girders, owing to the thickness of the web being small, compared with the depth of the girder, buckling of the web tends to take place in spite of stiffeners, and makes the condition of pure shear an impossibility: hence, the stiffeners must also act as struts; the shear stress of the material then approaches the value of $f/2$, and is equal to it when the web is wholly in tension and the struts under thrust. For this reason, then, the value $f/2$ errs on the side of safety, and is the more rational value to adopt; on the other hand, for girders which approach the type of rolled girders, in which the web itself acts as a stiffener or strut, a higher value can be adopted.

At first sight, this would appear to give marked economy for girders of this type; but the uneconomical distribution of the material required for stiffening the web as a strut in girders of any depth very quickly nullifies any advantage in this respect.

A web-plate with properly-designed struts will be stronger than a web-plate with stiffeners, when compared together under like conditions, the web-plate being thin compared with the depth of the girder. For short, shallow girders, this would not hold true, for the reason just mentioned, namely, that the web acts as a strut, and a higher value than $f/2$ can be adopted in ordinary cases; this limit seems to be when the depth of the girder is about two feet.

In the examination of old plate-girders with vertical struts, the line of wear of the rivet-holes in the web is at 45°; this is strong confirmatory evidence that the stress in the web is principally one of tension on lines at 45°, and that the struts are under compression. If the stress were one of pure shear, the wear would of necessity be along horizontal lines.

It follows, then, that plate-girders designed on the assumption that the web is in tension and the struts under thrust will err on the side of safety; because it has been shown (§ 45) that, if the web is designed on the ordinary theory of transmitting the shear stress alone, less material would be required. Owing, however, to the complexity of the problem, the material required for stiffeners does not admit of accurate solution, and the tendency is then to have an excess of material; whereas, in the other case, the problem is more determinate, and the stresses on the web and struts can be solved with a fair degree of accuracy. If, then, a girder is designed of sufficient strength on the lines of this analysis, it will be of more than sufficient strength when judged by the ordinary theory.

50. **Thickness of Web and Spacing of Struts.**—The shearing force at any section being known, the area of the cross-section of the web, or of the web and struts when inclined at 45°, or of the web when the struts are vertical, is

$$A = \frac{2F}{f},$$

if the material required for stiffening is neglected. Now, if the S. F. diagram is given, as in fig. 49, the shearing force at any section is known, being proportional to the ordinate drawn through the diagram at that section. From this it follows that, if a horizontal cross-section of the web, or web and struts, is taken, its area would vary point by point, similarly to the area of the S. F. diagram.

Now, imagine a plate-girder, as in fig. 44, with a number of ribs spaced close together on either side of the web, so as to form a ribbed plate: the stresses in such a girder would be much the same as that of fig. 38. If a horizontal section is taken as shown, the area of its cross-section would vary point by point similarly to the area of the S. F. diagram. This increase of area could be allowed for in two ways so far as the ribs or struts are concerned, either by increasing their width and thickness, or by spacing them closer together. Now, suppose the ribs are kept of constant section and the shearing force to vary, the number of ribs required at any section would then vary in exactly the same proportion, or, stated in other words, the number of ribs required per unit length along the girder would be proportional to the area of the shearing-force diagram per unit of length. From this it follows at once that, by dividing up the shearing-force diagram into a series of equal areas by lines perpendicular to the base, the required number and distribution of the struts are obtained. The shearing-force diagram on fig. 49 has been so divided, giving the distribution of the struts when of constant section, the struts being either vertical or inclined at 45°. The struts can, of course, be spaced in some other manner, as, for instance, at equal distances, the cross-section of the struts then varies proportionately to the intercepted areas of the shearing-force diagram. Both these cases are treated of in the design for the girder of § 56.

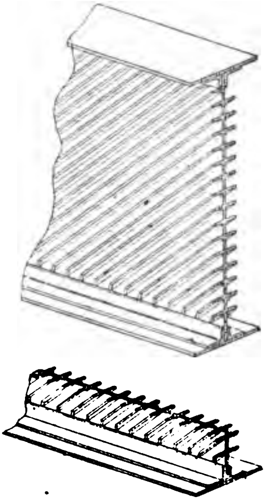

FIG. 44.

51. **Struts.**—To determine the dimensions of a strut to carry a given load, the Rankine-Gordon formula will be used, as it gives results which are most nearly in accordance with experiment.

The formula may be derived as follows:—Let a strut of length l, with round ends, carry a weight W (fig. 45). Under the action of the load the strut bends, the deflection being k. If the material is stressed to its working stress, k is the proof-deflection. For similar sections similarly loaded, $k \propto \frac{l^2}{y}$ (§ 35), where y is the depth of the section. Now, at the section at A the strut is transmitting the direct stress due to the load, and also that due to bending. Let $p'A = W$ where p' is the direct stress per unit of area, and A the area of the cross-section. Superimposed on this stress are the stresses due to bending.

The bending moment $M = Wk = p'Ak = \frac{pI}{y}$ (§ 31). Substitute for I its value $A\rho^2$ (§ 33); then $\frac{p}{p'} = \frac{ky}{\rho^2}$; but $k \propto \frac{l^2}{y}$, or $k = c\frac{l^2}{y}$, where c is some constant: hence,

$$\frac{p}{p'} = c\frac{l^2}{\rho^2}.$$

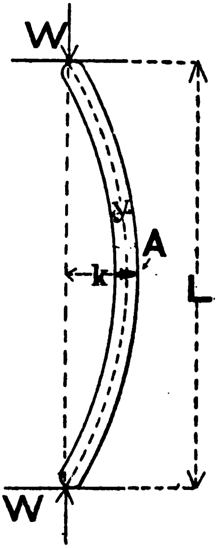

FIG. 45.

Add unity to both sides of this equation, then

$$\frac{p' + p}{p'} = 1 + c\frac{l^2}{\rho^2}.$$

Now, $p' + p$ equals the sum of the stresses due to the load and to bending, and is equal to f, the working strength of the material; also $p' = \frac{W}{A}$. Substituting these values, and reducing, the formula takes the form

$$W = \frac{fA}{1 + c\left(\frac{l}{\rho}\right)^2},$$

which is the Rankine-Gordon formula. For struts with round ends, $c = \frac{1}{9000}$; for fixed ends, $c = \frac{1}{36000}$; and for partially fixed ends, or for one end fixed, and one end round, $\frac{1}{20000}$; the value of f depending on the manner of loading of the strut (§ 9). Let $W = fA_1$, where A_1 represents the area required for a very short strut, and substitute in the above equation; then

$$\frac{A - A_1}{A_1} = c\left(\frac{l}{\rho}\right)^2.$$

Now let $A - A_1 = A_2$. This portion A_2 of the total area of the cross-section may be looked upon as that required to strengthen the strut against bending, and the other part of the area carries the load: hence,

$$\frac{A_2}{A_1} = c\left(\frac{l}{\rho}\right)^2.$$

Substitute for A_1 its value $\frac{W}{f}$; then $W = \frac{fA_2\rho^2}{cl^2}$; but $A_2\rho^2 = I_2$, the moment of inertia of that part of the cross-section due to bending, substituting $W = \frac{fI_2}{cl^2}$. Compare this with Eüler's well-known formula for struts, viz., $W = \frac{\pi^2 EI_2}{l^2}$, where E is Young's modulus, the other terms having the same significations, the factor $\pi^2 E$ taking the place of $\frac{f}{c}$, which is a constant. Now, I_2 does not take into account the area required to support the load, and this area can only be neglected when it is small compared with the area required for bending, and this only occurs in long struts: hence, Eüler's formula only applies to such cases.

52. **Equivalent Length of Booms.**—For the design of the booms of girders, it is not only necessary to determine their sectional areas to resist the direct stresses under thrust and tension: the top boom under thrust must, in addition, be strong enough to resist bending as a strut. For the majority of girders, a breadth of $\frac{1}{30}$ to $\frac{1}{40}$ the span is commonly used for the booms; and this has been found in practice to give the necessary stiffness.

Suppose the girder loaded with a uniform load over the whole span—this being the distribution of the load which gives the maximum stress (§ 24). The B. M. and S. F. diagrams will be as in figs. 9, 10. The thrust on the boom at any section is proportional to the ordinates of the B. M. diagram, the depth of the girder being constant; or, in other words, to the area of the S. F. diagram, since $M = \int F dx$.

If the S. F. diagram is divided up into a number of small areas, and loads proportional to these areas be applied to a number of struts whose lengths are twice the distance of these areas from the centre of the boom, then the determination of the sectional area of the boom, considered as a strut, would resolve itself into the determination of the sectional area required by the sum of a number of struts carrying the varying loads.

Using the Rankine-Gordon formula in the form

$$W\left\{1 + c\left(\frac{l}{\rho}\right)^2\right\} = fA,$$

and keeping the radius of gyration constant for all values of the lengths, then the required summation would be of the form

$$W_0 + \frac{c}{\rho^2} W_0 l_0^2 = fA_0,$$

$$W_1 + \frac{c}{\rho^2} W_1 l_1^2 = fA_1,$$

$$W_2 + \frac{c}{\rho^2} W_2 l_2^2 = fA_2, \text{ etc.,}$$

$$\Sigma W + \frac{c}{\rho^2} \Sigma W l^2 = f \Sigma A;$$

but $\Sigma W = Q$, the thrust at the centre of the boom, and $\Sigma A =$ the area of the section. It only remains to determine the value of the $\Sigma W l^2$, which can be done as follows:—

Let ydx be the small thrust on the length of the strut $2x$; then the required summation can be expressed as

$$4 \int x^2 y dx \qquad \text{and } y = ax,$$

$$= 4a \int x^3 dx$$

$$= \Big[ax^4 \Big]_0^{\frac{l}{2}} \qquad \text{but from the S. F. diagram}$$

$$= \frac{Ql^2}{2}: \qquad Q = \frac{ax^2}{2}, \text{ and } x^2 = \frac{l^2}{4}$$

hence the equivalent length $= \dfrac{l}{\sqrt{2}} = \cdot 7l.$

Hence, in designing the boom as a strut, it is sufficient to consider it of the equivalent length $\cdot 7l$ under a thrust Q.

An allowance for rivet-holes must be made when determining the gross sectional area of the boom under tension; for the boom under thrust, no such deduction is necessary. As the rivets fill the holes, and help to transmit the thrust, it will be found when this is done, that with a breadth of boom of $\frac{1}{30}$ to $\frac{1}{40}$, the span, the extra material required to resist bending as a strut is, as a rule, small, and for short spans can be neglected altogether.

53. **Live Loads on Railway Under-bridges.**—These depend upon the locomotives and rolling-stock for any particular railway, the equivalent uniformly-distributed loads for a single track adopted by the various British railways showing considerable variation. On the diagram of fig. 46, the values adopted by seven of these, as given by Conradi,* are shown; and also to these there have been added the moving loads adopted by Farr.† To allow for the extra stresses due (*a*) to impact owing to the load being suddenly applied, (*b*) to the badly-laid or maintained state of the permanent way, (*c*) to badly-balanced locomotives, (*d*) to the angle of obliquity of the connecting-rods of the locomotives, Farr adds a percentage to the moving load; this is shown by the lower dotted line on the diagram: the addition of the ordinates of this curve to those of the moving load gives the curve for the maximum load, with percentage for impact, as shown by the upper thick line. The moving loads, as given by Farr, are the maximum loads obtained from the comparison of a large number of locomotives in use on British railways; no standard of loads having been yet adopted, they are specially useful for the purposes of design.

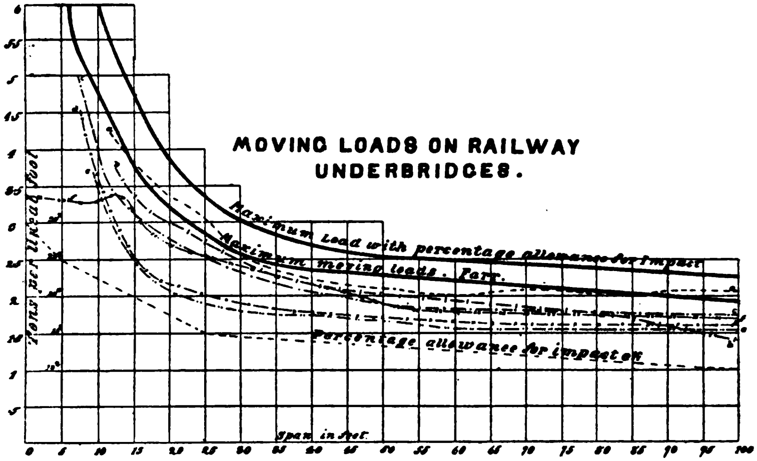

FIG. 46.

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* *Engineer*, Oct. 18, 1895.
† Proceedings I. C. E., vol. cxli.

The moving loads on the cross-girders are shown on the diagram of fig. 47, together with the allowance for impact, the allowance for impact in this case being greater than for the main girder. This seems reasonable, as the floors and cross-girders yield relatively more than the main girders, and are more affected by the periodic stresses due to this cause.

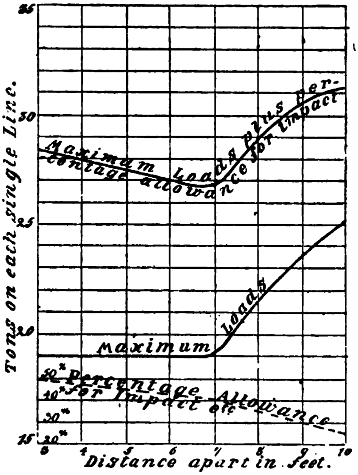


FIG. 47.

The following formulæ, which have been deduced by the author from Farr's paper, give, with considerable accuracy, the maximum values contained therein :—

The equivalent moving load in tons per foot

$$= \frac{30}{\text{span in feet}} + 1{\cdot}75$$

With the percentage allowed for impact by Farr, the formula is—The equivalent moving load in tons per foot

$$= \frac{41}{\text{span in feet}} + 1{\cdot}75.$$

These formulæ show that the equivalent moving load depends upon a constant which is the equivalent moving load over a span of maximum length, together with a constant load divided by the span.

The whole question of the equivalent moving load can therefore always be reduced to an equation of the form

$$y = \frac{A}{x} + B,$$

the value of the constants A and B depending only on the rolling stock for any particular railway.

54. **Live Loads on Road-bridges.**—The maximum weight per square foot of a crowd is given by Stoney as 147 lb.; according to Fidler this appears excessive: he

recommends 1 cwt. for bridges open to the public near busy thoroughfares.

Provision also requires to be made for loads such as traction-engines and the like, the concentrated loads on the wheels of which may vary from 5 to 10 tons. Such loads may produce great local stresses; and, in designing such bridges, rolling and uniformly-distributed loads should be taken into consideration.

55. **Dead Load.**—For road-bridges, this is subject to so great variations that definite data are not available; for railway under-bridges, the weight of flooring and wind-bracing for a double track resting on two main girders is estimated by Sir B. Baker as follows:—

Span in feet.	Cwt. per foot-run.
10–100	14
100–150	15
150–200	16
200–250	17
250–300	18

The above loads to be reduced 2 cwts. per foot-run when the double track is supported on three main girders, and by 4 cwts. when supported on four main girders. The total dead load is made up of the main girders, floor, and cross-girders (if any), together with the permanent way, ballast, and concrete. Some examples from actual practice as given by Couradi are quoted in § 64, and others in § 69.

56. **Design for a Plate-Girder.**—As an illustration of the foregoing analysis, a design for a plate-girder is shown on figs. 52 and 53. The following are the required data:—

Bridge to carry double line of railway on two main girders.

Moving load reduced to an equivalent dead load of 2 tons per foot.

Dead load for one girder and one-half the total flooring-rails and ballast = ·8 ton per foot.

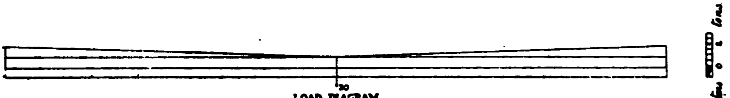
Clear span, 60′ 0″; depth, 7′ 6″.

The material of the girder to be of mild steel, having a breaking-tensile strength of 30 tons per square inch.

A factor of safety of 3 will be taken, and the stresses determined in accordance with Wöhler's law: the formula of § 9 is

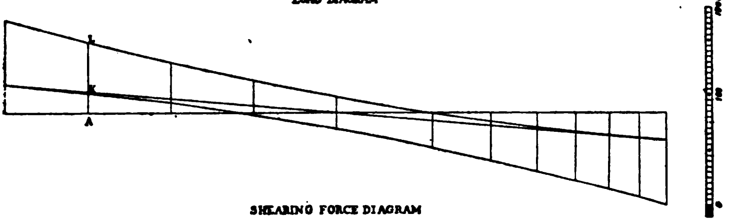
$$\text{working strength} = f = \frac{5}{1 - \frac{1}{2}\frac{\min S}{\max S}}.$$

The L., S. F., and B. M. diagrams are shown on figs. 48, 49, 50, 51; the load diagram refers to that distribution of load



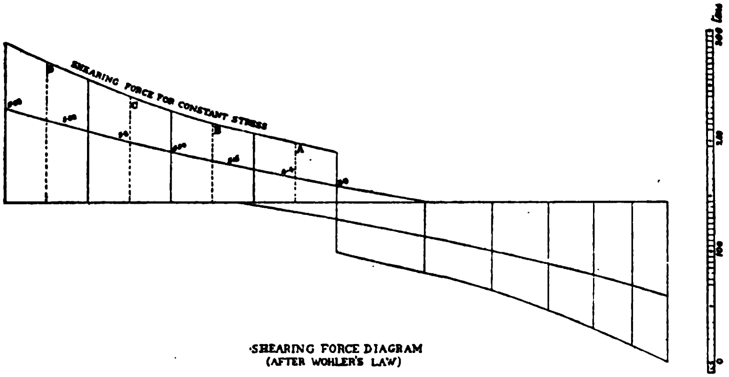
LOAD DIAGRAM

FIG. 48.



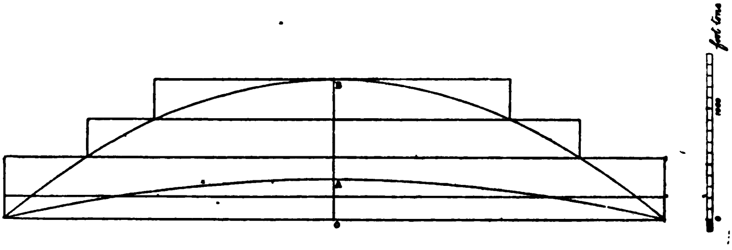
SHEARING FORCE DIAGRAM

FIG. 49.



SHEARING FORCE DIAGRAM
(AFTER WOHLER'S LAW)

FIG. 50.



BENDING MOMENT DIAGRAM—LOAD UNIFORMLY DISTRIBUTED

FIG. 51.

which would give the S. F. diagram for the moving and dead loads (§ 26). For the B. M. diagram, the loads are taken as uniformly distributed.

To determine the working stress for the web at various sections, the ratio of the $\frac{\min S}{\max S}$ must be scaled from the diagram, the ratio at any section being the ratio of the ordinates of the S. F. diagram for the dead load, and for the live and dead loads together, such as $\frac{AK}{AL}$; at the ends of the span, the values are

$$\frac{\frac{1}{2}\text{dead load}}{\frac{1}{2}(\text{live} + \text{dead loads})} = \frac{\min S}{\max S} = \frac{24}{84} = \frac{2}{7},$$

and

$$f = \frac{5}{1 - \frac{1}{2}\frac{\min S}{\max S}} = \frac{5}{1 - \frac{1}{2} \times \frac{2}{7}} = 5\cdot 8 \text{ tons.}$$

The values of the working stress for a number of sections are marked on the lower diagram (fig. 50).

The area of the cross-section of the web (§ 45) is

$$A = \frac{2F}{f}.$$

Since the depth of the girder is constant,

$$A = Dt, \quad \text{or} \quad t = \frac{2F}{Df},$$

the values of the working stress and the shearing forces at the several sections being known, the thicknesses of the web can be determined.

It is more convenient to proceed as follows:—Consider what S. F. diagram would give the same areas for the different sections under a constant working stress of 10 tons, the stress for a steady load; that is, put

$$A = \frac{2F}{f} = \frac{2F'}{10}, \quad \text{or} \quad F' = \frac{10F}{f}.$$

If several values are plotted and a curve drawn through them, a new S. F. diagram is obtained which, on the hypothesis of a constant stress of 10 tons, gives the same values for the areas of the various sections as would be obtained from the calculations direct. The upper curve of the S. F. diagram of fig. 50 has been thus obtained; the ordinates of this diagram are proportional to the thicknesses of the web at the various sections.

The scaled values of F' at the sections A, B, C, D are 54, 71, 95, and 127 tons, respectively, the thickness of the web $t = \frac{2F'}{10}$ giving values of ·12″, ·16″, ·21″, and ·28″, at the respective sections. At A and B the thicknesses are so thin, that no practical use can be made of them; and the web will be made $\frac{1}{4}$″ thick, this being the least value of the thickness adopted in practice. To allow for rivet-holes and wear, the thickness of the web at D has been increased by about 25 per cent., giving a thickness of $\frac{3}{8}$″; and at C a thickness of $\frac{5}{16}$″ has been adopted.

The B. M. diagram is shown on fig. 51: the curve passing through A is the B. M. diagram for the dead load alone, and through B the B. M. diagram for the live and dead loads combined.

Now, OA and OB to scale are the bending moments at the centre of the span: hence

$$OA = M_1 = \frac{WL}{8} = \frac{48 \times 60}{8} = 360 \text{ ft.-tons};$$

$$OB = M_2 = \frac{(W+U)L}{8} = \frac{168 \times 60}{8} = 1260 \text{ ft.-tons}.$$

Since the depth of the girder is constant, the stress in the booms will be proportional to the bending moments, and the ratio of the

$$\frac{\min S}{\max S} \text{ will be } \frac{OA}{OB} = \frac{M_1}{M_2} = \frac{360}{1260} = \frac{2}{7}.$$

Putting this value in the formula for the working stress,

$$f = \frac{5}{1 - \frac{1}{2}\frac{\min S}{\max S}} = \frac{5}{1 - \frac{1}{2} \cdot \frac{2}{7}} = 5{\cdot}8 \text{ tons}.$$

To determine the sectional area of the boom, from § 33:

$$M = fAh = fAD,$$

where A = the area of the cross-section, and $h = D$ = the depth of the girder. Putting in the above values,

$$A = \frac{M}{fD} = \frac{1260}{5{\cdot}8 \times 7{\cdot}5} = 29 \text{ square inches, nearly.}$$

Let the breadth of the booms be $\frac{1}{36}$ of the span = 1′ 8″, and built up of $\frac{1}{2}$″ plates and two 4″ × 4″ × $\frac{1}{2}$″ angle-irons; then the gross sectional area as designed gives 38 square inches, and, after allowing for rivets, this reduces to 31 square inches. The horizontal lines on the B. M. diagram are marked off at distances proportional to the area of the angle-irons and plates, respectively, and the lengths of the plates are determined from the lengths of these lines.

The sectional area of the boom, considered as a strut, will be determined from the Rankine-Gordon formula (§ 51), using it in the form

$$A_2 = A_1 c \left(\frac{l}{\rho}\right)^2.$$

Now, $A_1 = 29$ square inches, and the equivalent

$$l = \cdot 7L = \cdot 7 \times 60 = 42' \quad (\S\ 52);$$

ρ, for the section as designed, is nearly 6″; and for c, the value $\frac{1}{20000}$ will be used, as the boom may be considered to be partially fixed at the ends: hence,

$$A_2 = \frac{29 \times 42^2 \times 12^2}{20000 \times 6^2} = 10 \text{ square inches, nearly,}$$

and

$$A_2 + A_1 = 29 + 10 = 39 \text{ square inches.}$$

Now, the area as designed is 38 square inches—a sufficiently close approximation.

Some provision must be allowed for the bending action of the booms round the ends of the struts, as shown on figs. 37, 39. By assuming the boom to be a continuous girder with the struts for the points of support, and loaded with the vertical components of the stresses in the web, an approximate calculation can be made of the bending moments; the values, however, are very small, and can easily be provided for by adding to the depth of the flanges of the angle-irons connecting the web to the booms, or by a small addition to the web itself: in the design, a small allowance has been made in both these directions.

The design of the struts will now be considered. Two arrangements for these are shown on figs. 52 and 53—one with the struts spaced equally and of varying section, and the other with the struts spaced unequally and of constant section. The loads on the equally-spaced struts are the vertical components of the stresses acting in the web, and will have the same values as already used in designing the web, viz., 54, 71, 95, and 127 tons; the additional portion of the shearing force as scaled from the ordinates produced through the struts on the S. F. diagram is transmitted by the web, and does not come on that particular strut.

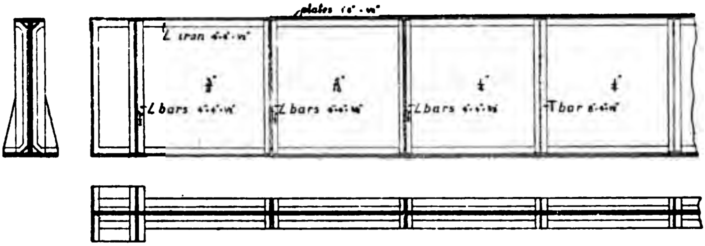


FIG. 52.

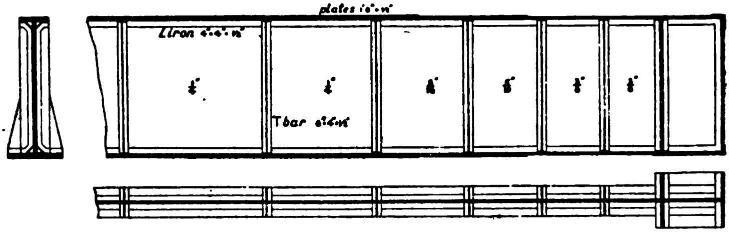


FIG. 53.

Using the Rankine-Gordon formula for struts in the form

$$A_2 = A_1 c \left(\frac{l}{\rho}\right)^2,$$

and adopting a value of

$$\left(\frac{l}{\rho}\right) = 40, \quad \text{and for} \quad c = \frac{1}{9000},$$

that is, assuming the struts to have round ends, the formula takes the form

$$A_2 = \cdot 2A_1, \quad \text{or} \quad A = A_2 + A_1 = 1{\cdot}2A_1.$$

Now, $A_1 = \dfrac{F'}{f}$ (§ 47), and f the working stress = 10 tons, because in the scaled value of F' from the S. F. diagram allowance was made for the working stress in accordance with Wöhler's law: hence,

$$A_1 = \frac{F'}{10}, \quad \text{and} \quad A = \cdot 12F'.$$

So long as the ratio $\frac{l}{\rho} \leqq 40$, the column will be as strong as, or stronger than, is necessary. The depth of the girder is 7′ 6″ = 90″, and therefore

$$\rho = \frac{l}{40} = 2\tfrac{1}{4}''.$$

To design the struts, the area of the cross-section A is determined; a sketch is then made of the desired type of cross-section, having an area of A square inches. From this sketch, the value of ρ can, with some practice, be determined approximately by inspection, and afterwards, when necessary, checked by calculation (§ 32). The struts shown on fig. 52 have been designed on these lines.

The disposition of the struts, where the cross-section is constant, is obtained from the equal areas shown on the S.F. diagram. The first strut is designed to carry the load equal to the shearing force intercepted on the S. F. diagram from the first strut; in this case, the same as for section A, the equal areas then give the number and position of the remaining struts.

In practice, the disposition of the struts would require to be slightly varied to suit the cross-girders or troughing that might be adopted; by making use of the analysis, the disposition is, in all cases, a simple matter.

57. **Rivets.**—The breaking-strength to shear of steel rivets was given in § 6 as 20–25 tons per square inch, or nearly $\frac{3}{4}$ the strength to tension, pure shear, or compression; the working strength then to shear can be taken as $\frac{3}{4}$ that to tension, that is, $f_s = \frac{3}{4}f$. In calculating the area required for rivets, this value will be used. The pressure of the plate on the outside of the rivet should not exceed 10 tons per square inch, and a value of 8 tons is commonly adopted. Interpreted by Wöhler's law, the load per square inch would be about 15 tons; that is, one-half the compressive or tensile strength, giving a factor of safety of 2, and a bearing-pressure of 8 to 9 tons per square inch in the usual run of bridge-work.

Under such loads, it is essential that the rivets should fill the holes in the plates and connections, as any looseness of the rivets would give rise to bearing-pressures much greater than the above. The bearing-area of the plate on the rivet is taken as the diameter d of the rivet multiplied by the thickness t of the plate or bearing-area = dt.

In designing joints and connections in plate-girders, provision must be made in the rivets for shearing area and bearing area.

The rivets required for connecting the web to the booms of the girder in figs. 52, 53 will now be considered. It was shown in § 39 that the shear stress along the flanges per unit of area on any section is the same as that per unit of area on a vertical cross-section of the web at that section: hence, if the shear stress at a vertical section is known, the shear stress on the rivets is also known. Usually, in designing plate-girders, the web is assumed to be under a pure shear stress, and the rivets are designed accordingly. Considered, however, from the standpoint of this analysis, the number of rivets so determined would not be sufficient; and there seems to be little doubt that the rivets in the majority of plate-girders get more work to do than what they are designed for.

Consider the section (D) on the S. F. diagram of fig. 50: the shearing stress F' is transmitted by the tension of the web to the strut at the end of the span. Now, the breadth of the plate transmitting this stress is $\dfrac{D}{\sqrt{2}}$; and the total tension on this breadth

$$= T = \frac{fDt}{\sqrt{2}}; \text{ but } F' = \frac{fDt}{2} \text{ (§ 45): hence, } T = \sqrt{2}F'.$$

This tension T is distributed along a length of boom equal to the depth; and the shear stress is in the proportion of $\sqrt{2}$ to 1 when compared with the assumption of a pure shear stress. The value of F' is 127 tons. As Wöhler's law is allowed for in this value, a factor of safety of 3 will be as true for the rivets as for any other part of the girder: hence

$$f_s = \frac{3 \times 10}{4} = 7{\cdot}5 \text{ tons.}$$

The diameter of the rivets:—For anything under $\frac{1}{2}''$ plates, $\frac{3}{4}''$ diameter rivets are usual, and for $\frac{1}{2}''$ to $\frac{3}{4}''$ plates, $\frac{7}{8}''$ diameter. A convenient rule for determining the diameter for rivets in terms of the thickness t of the plates is

$$d = 1{\cdot}2\sqrt{t}.$$

Since the web at the section is $\frac{3}{8}''$ thick, the diameter of the rivets would be $\frac{3}{4}''$, and the number n required will be

$$2nf_s a = \sqrt{2}F',$$

where $f_s = 7{\cdot}5$ tons, a the area of a $\frac{3}{4}''$ rivet $= {\cdot}44$ square inches, and the coefficient 2 is used because the rivets are in double shear: hence,

$$n = \frac{\sqrt{2}F'}{2f_s a} = \frac{\sqrt{2} \times 127}{2 \times 7{\cdot}5 \times {\cdot}44} = 27.$$

The bearing area equals

$$ndt = 27 \times \tfrac{3}{4} \times \tfrac{3}{8} = 7{\cdot}6 \text{ square inches.}$$

This multiplied by 15 = 114 tons, and it should be 127, this being the value of F' by Wöhler's law. The bearing area is not sufficient; and one of the following must be done:—Either put in more rivets of a smaller diameter, or more rivets of the same diameter, and thus have more shearing area than is necessary, or increase the diameter of the rivets. It will be found that 30 rivets, $\frac{3}{4}''$ diameter, would be sufficient, or about 24 rivets, $\frac{7}{8}''$ diameter. The pitch of the rivets in this latter case would be $\frac{90}{24}$, or nearly 4″, and in the former $\frac{90}{30}$, or 3″, which is close, and reduces efficiency of joint. The $\frac{7}{8}''$ diameter with 4″ pitch would be the best, as it would not be advisable to reduce the rivets below $\frac{3}{4}''$ diameter; the thickness of the angle-iron flanges being $\frac{1}{2}''$, a $\frac{3}{4}''$ or $\frac{7}{8}''$ rivet would be more suitable. The pitch and diameter of the rivets being determined for one section, the number required at any other section will be directly proportional to the shearing force at that section, so long as the plate remains of the same thickness.

In addition to determining the pitch, diameter, and bearing-area of the rivets, the *efficiency* of the joint requires to be considered. An addition of 25 per cent. to the web was made, giving a thickness of $\frac{3}{8}''$; this means that the joint is supposed to have an efficiency of 80 per cent., or that the area of the plate between the rivets is to the area cut out by the diameter of the rivet in the proportion of 4 : 1. Now, a $3\frac{3}{4}''$ or 4″ pitch is commonly used with $\frac{3}{4}''$ rivets; and the proportion would be $3 : \frac{3}{4}$ in the case of $3\frac{3}{4}''$ pitch, that is, 4 : 1; or with 4″ pitch, $4\frac{1}{4} : 1$; any pitch then less than $3\frac{3}{4}''$ would put higher shearing stresses on the web-plate between the rivets than was intended. For reasons already pointed out above, this is sometimes done, and a pitch of 3″ is not uncommon.

The joints in the web have not been shown, as these depend upon the sizes of the plates. Some types of these joints are shown on fig. 54. What has been remarked for the riveting of the web to the flanges applies equally to these joints.

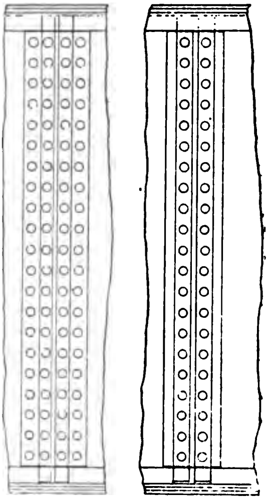


Fig. 54.

58. **The Web under Thrust.**—The girder thus designed is stronger than if it had been designed on the assumption of pure shear stress (§ 49). It may be argued that the web in such a girder does transmit some thrust; approximately it may be calculated thus:—The web-plate at the end panel of the girder on fig. 52 is $\frac{3}{8}''$ thick, and forms a square. Consider it under thrust, and assume that the average length of the web, considered as a strut, is equal to the depth of the girder $= 90''$, that is, nearly ·7 the length of the diagonal, a similar investigation to that of § 52 giving this value. The edges of the web-plate are fixed: hence, the plate may be considered as a strut, with fixed ends, and the load carried per inch of breadth of the plate will be (§ 51),

$$W = \frac{fA}{1 + c\left(\frac{l}{\rho}\right)^2} = \frac{5{\cdot}5 \times \frac{3}{8}}{1 + \frac{12 \times 90^2}{36000 \times \left(\frac{3}{8}\right)^2}} = \frac{1}{10} \text{ ton nearly.}$$

Now, $f = 5{\cdot}5$ tons, $A = bt$, and $b = 1''$, and $t = \frac{3}{8}''$, $c = \frac{1}{36000}$, $\rho^2 = \frac{t^2}{12}$ (§ 33), and $l = 90''$, putting in these values.

The load on a breadth of $90'' \times \sqrt{2} = 126'' = 12{\cdot}6$ tons, or approximately $\frac{1}{10}$ the shear stress at the section; this, if allowed for, would slightly reduce the material in the struts about $\frac{1}{10}$, a quantity small enough to be neglected.

59. **Wind-Pressure.**—The Board of Trade recommendations for wind-pressure are as follows:—

(1). That in exposed situations the maximum pressure to be provided for shall be 56 lb. per square foot of surface.

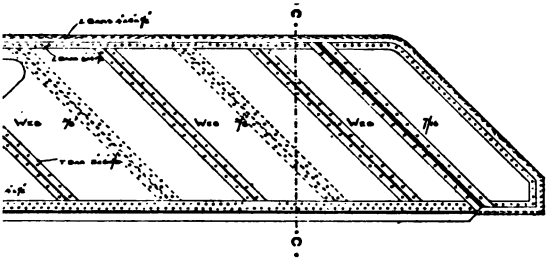
(2). That for open lattice-work the surface on which this pressure acts should be from once to twice the front area according to the openings in the lattices.

(3). That for iron or steel work, a factor of safety of four should be provided; and considering the tendency of the bridge as a whole to be overturned, a factor of safety of two should be allowed. There is a considerable divergence of opinion as to the maximum pressure per square foot that should be adopted, experiments having shown that the pressure varies greatly over large areas, and that the average maximum pressure is not more than 70 per cent. of what it would be over a small area of about 1 square foot; for this reason some engineers recommend a pressure of 45 lb. per square foot over the whole exposed surface.

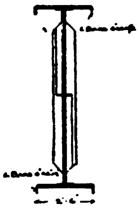
In the design of the girders of §§ 56, 61, the effect of wind-pressure has been neglected. To consider it, requires that the normal load due to the wind-pressure acting on the girder should be given, and that the girders should be made strong enough to transmit the bending moments due to this normal load to the cross-girders and troughing. Additional material requires to be added to the struts for this purpose, usually in the form of gusset-plates connecting the cross-girders or troughing to the struts at intervals along the girder. For small girders, the effect of wind-pressure is negligible; and for moderate spans, the additional material required is small. For large spans, wind-bracing is adopted, and reduces the material that would otherwise be required for the struts and connections. Owing to the uncertainty of the data with regard to wind-pressure and its effect on girders of large span, the disposition of the material to resist it is largely a matter of experience.

60. **Stiffening against Distortion.**—In addition to an allowance of material for wind-pressure, the stiffening of the ends to prevent distortion of the girder sideways requires to be considered. The allowance for this depends on the strength of the booms to take up the small bending and twisting moments introduced by the stresses not acting in a vertical plane—the booms, together with the ends of the girders, tending to move relatively to one another, and form a sort of exaggerated figure 8.

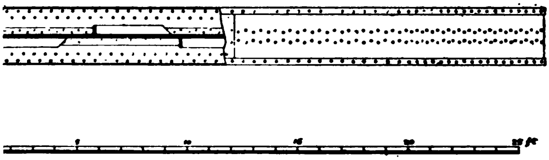
Suppose the flanges connected to the struts and web by ball-joints, then, although the girder would be of sufficient strength to transmit the stresses in a vertical plane, it would be unstable; the gusset-plates prevent this relative movement of the flanges by introducing moments which give rise to torsion of the flanges.

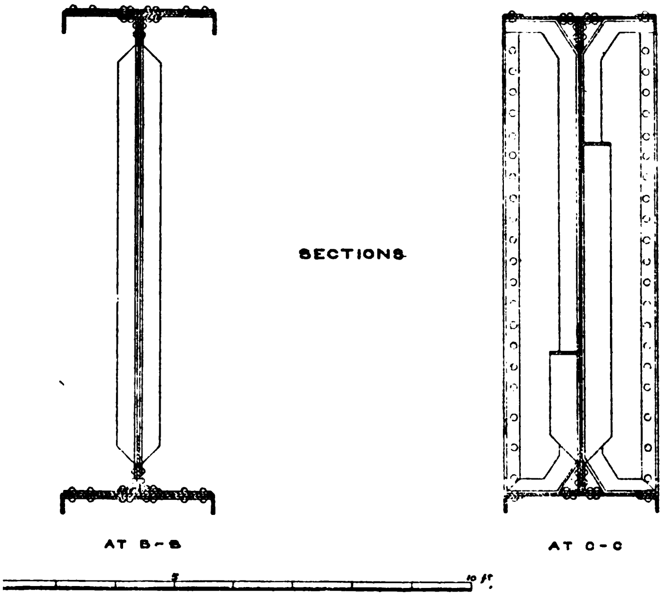


STRUTS EQUALLY LOADED



SECTION





SECTIONS

AT B-B

AT C-C

The disposition of the extra material in the gusset-plates for this purpose is a matter of experience, and has not been allowed for in the design of fig. 52.

In small girders allowance is made for this by bringing one plate of the booms together with the angle-irons round the ends of the girder, to which the connections are strongly made, and in addition, in larger girders gusset-plates connect the flanges and flooring at intervals along the girder, which thus play a double part in transmitting the stresses due to wind-pressure, and at the same time stiffen the girder against distortion: the deeper the girder relatively to the span, the more necessary is it to make provision against the girder distorting in this manner.

61. **Design for a Compound Girder 80′ 0″ Span.**—It was pointed out in § 47, that there would be an economy of material in placing the struts at an angle of 45° instead of vertical; also, the analysis given in §§ 65–74 shows that 45° bracing gives maximum economy. For small spans up to about 60′ 0″ the saving would not be sufficient to balance the practical advantages; with longer spans, this would not hold true, and the design shown on fig. 55 has been prepared to show this.* Full particulars of the required data are given on the drawing, together with the L., S. F., and B. M. diagrams.

The shearing stress being comparatively small for about $\frac{1}{3}$ the span, the web has been discarded altogether, and a light cross-bracing substituted, thus effecting a saving of material in this part of the girder.

Two arrangements for the struts have been shown, as in the previous design of § 56—one in which they are spaced equally and of varying section; the other in which the cross-section of the struts is constant, and the spacing variable. The section of the web and struts at any cross-section is proportional to the shearing force at that section, being equally divided between them—additional material being added for the stiffening of the struts against bending.

Nothing has been allowed for connections between the booms, cross-girders, or troughing to allow for wind-pressure, distortion, etc., it being deemed advisable to limit the design to stresses acting in a vertical plane only.

Girders of this type admit of being designed for large spans, and have a slight advantage in economy of material compared with latticed or triangulated girders, and the cost of construction would be considerably less. The connection of the cross-girders or troughing direct to the web is an advantage compared with the channelled cross-section of the booms in latticed and triangulated girders.

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* Minutes of the Institution of Civil Engineers of Ireland, 1903.

62. **Design for a Lattice-Girder.**—The design given on fig. 56 is intended for girders of and above 120′ 0″ span: it illustrates the practical application of utilising ties and struts of constant section throughout for the lattice-bracing, the spacing being determined from the equal areas of the S. F. diagram of § 50. Such a girder would require but few different sections throughout for its construction, and, at the same time, would be equally economical of material when compared with the triangulated girder under like conditions, and the cost of construction would be considerably less.

A double system of lattice-bracing has been shown. By reducing this to a single system the girder approaches the type of a triangulated girder, the number of ties and struts being few more than would be required in a triangulated girder with a similar system of equidistant triangulation.

From the analysis given in §§ 70–74, the design gives maximum economy of material; other alternative systems of bracing are shown on fig. 69, Nos. 1, 2, 3.

63. **Depth of Girders.**—The economic depth in plate-girders of small span is about $\frac{1}{10}$ the span, increasing to about $\frac{1}{8}$ span in girder of and above 80′ 0″ span. Various considerations affect the depth to be chosen, depending upon the conditions and situation for which the girders are required; the above values are those usually given as deduced by experience. For the type of compound girders as given on fig. 56, it is probable that the economic depth is not less than $\frac{1}{8}$ the span; and for girders of long span, carrying heavy loads, the depth could, with advantage, be increased beyond this value.

DESIGN FOR A LATTICE GIRDER

STRUTS AND TIES OF CONSTANT SECTION WITH VARIABLE SPACING

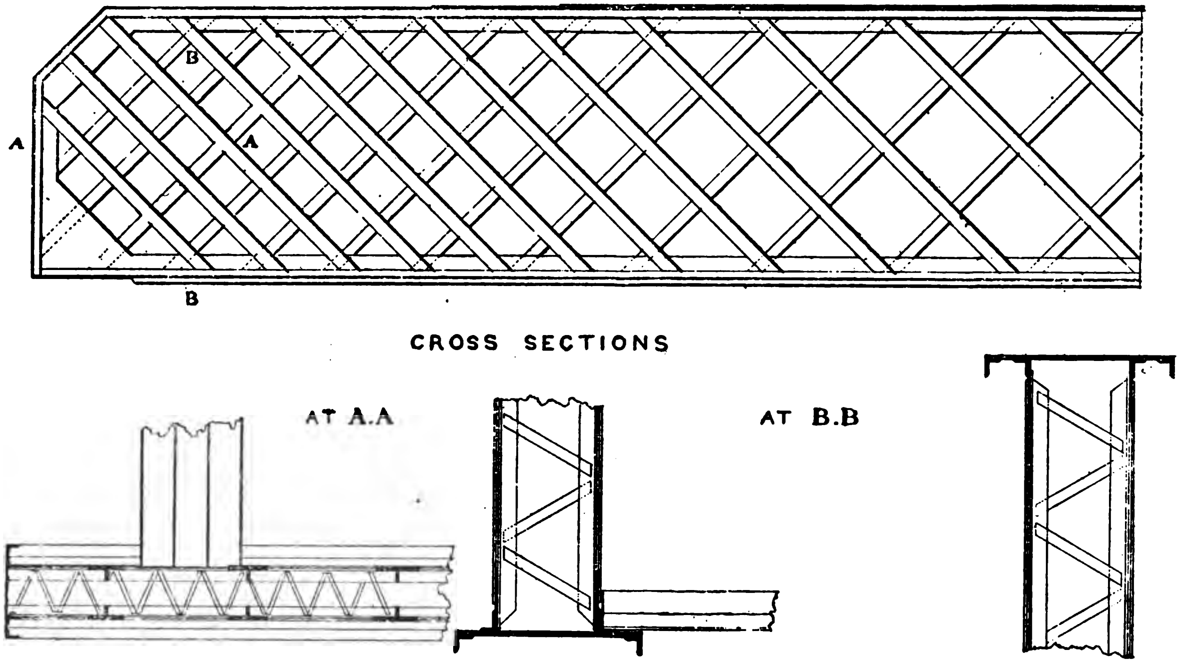

Fig. 56.

STANDARD TYPES OF SUPERSTRUCTURE.

64. The following sketches and description of types of superstructure are extracted by permission from a paper by Mr. J. P. Couradi*:—

"TYPE 1.—The most general type of superstructure, and adopted always for any spans where headway will admit, is to have the main girders directly under the rails. This system finds most favour amongst railway engineers, and is by far the most economical. It is also most important that there should be a continuous support directly under the rail. In all types submitted, this will be found a ruling feature, forming what are termed rail-girders, rail-trough girders, and rail-bearers. Another essential element of bridge-design which should be always aimed at, wherever possible in any kind of superstructure, is what is termed a free sleeper-road. This is a road which keeps the permanent way uniform and continuous throughout. The road can then be easily got at for very obvious reasons; this of course finds most favour amongst permanent-way inspectors. In a fixed sleeper-road, the sleepers are usually laid longitudinally, called a longitudinal way-beam, but of larger section than the ordinary transverse sleeper; the way-beams are fixed to the cross-girders by means of channels or angles. The discontinuity of this with the ordinary road is always a great disadvantage. Another very important point is that the superstructure should be kept watertight. In all the designs submitted it will be noticed that the road is laid in a trough-formed floor, consisting of flat floor-plates forming the bottom. The fender-plates form the sides of the trough. The floor-plates are not necessarily flat, but may be of the dished form, or Mallet's buckled-plate flooring, or any of the recognized forms of flat floorings. The fender-plates run longitudinally as shown in types 1 to 6, and are fixed to the floor by means of angles. It will be seen that the floors are watertight; provision, however, should be made for drainage. An initial camber given to the main girders secures this, the floor draining both ways, and at the ends the floor-plates are curved over downwards, thus giving additional means of drainage. The road is first laid with 3″ of

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* *Engineer*, October 18, 1895.

TYPES OF RAILWAY BRIDGE FLOOR STRUCTURES

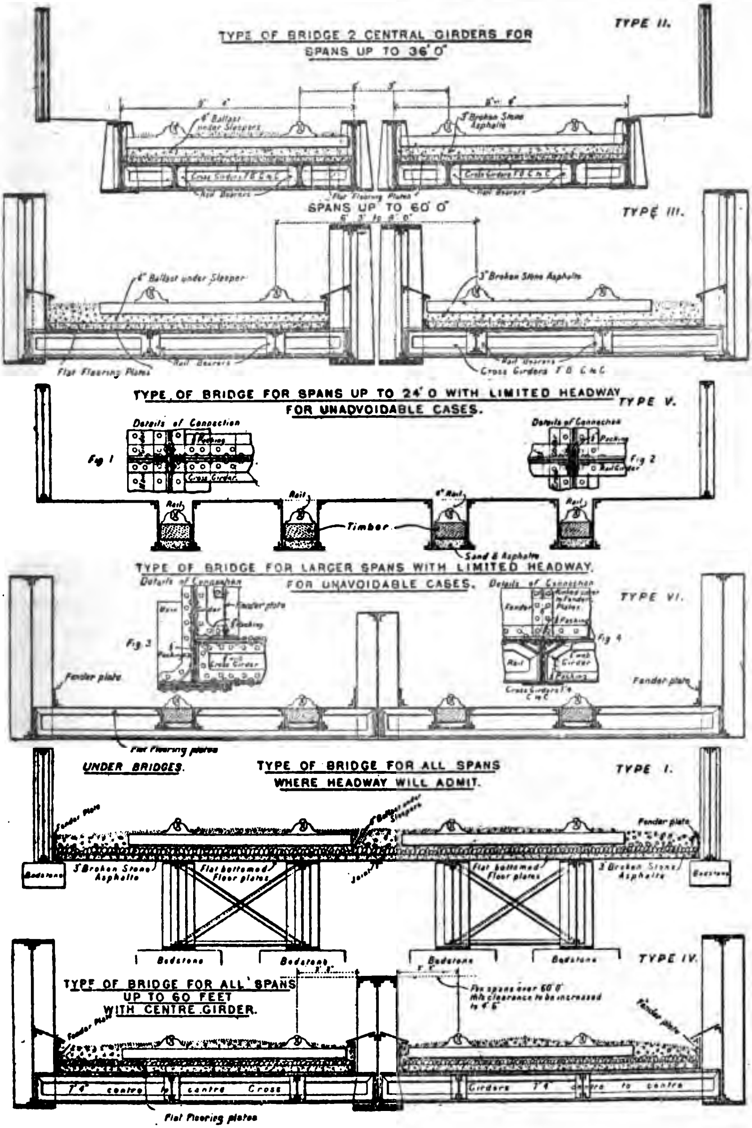

TYPE OF BRIDGE 2 CENTRAL GIRDERS FOR SPANS UP TO 36' 0"
TYPE II.
4" Ballast under Sleepers
3" Broken Stone Asphalte
Cross Girders 7' 6" C to C
Rail Bearers
Flat Flooring Plates
SPANS UP TO 60' 0"
TYPE III.
4" Ballast under Sleeper
3" Broken Stone Asphalte
Flat Flooring Plates
Rail Bearers
Cross Girders 7' 6" C to C
TYPE OF BRIDGE FOR SPANS UP TO 24' 0 WITH LIMITED HEADWAY FOR UNADVOIDABLE CASES.
TYPE V.
Details of Connection
Fig 1
Packing
Cross Girder
Fig 2
Rail Girder
Rail
Timber
Sand & Asphalte
TYPE OF BRIDGE FOR LARGER SPANS WITH LIMITED HEADWAY. FOR UNAVOIDABLE CASES.
TYPE VI.
Fig 3
Fender plate
Packing
Cross Girder
Fig 4
Fender
Riveted under to Fender Plates
Rail
Cross Girders 7' 6" C to C
Flat Flooring plates
UNDER BRIDGES.
TYPE OF BRIDGE FOR ALL SPANS WHERE HEADWAY WILL ADMIT.
TYPE I.
Fender Plate
Ballast under Sleepers
3" Broken Stone Asphalte
Flat bottomed Floor plates
Joint
Bedstone
TYPE OF BRIDGE FOR ALL SPANS UP TO 60 FEET WITH CENTRE GIRDER.
TYPE IV.
For spans over 60' 0" this clearance to be increased to 4' 6"
Fender plate
7' 4" centre to centre Cross Girders 7' 4" centre to centre
Flat Flooring plates

Fig. 57.

broken-stone asphalte, being an additional safeguard for a water-tight floor, and protects the floor from corrosion, and gives a good bed for the ballast. Under the sleepers, and immediately above the asphalte, 4″ at least of picked ballast is laid, thus securing a good bed for the sleepers, and keeping the asphalte free from vibrations causing cracking of the same. The main girders are braced underneath against wind-pressure, securing a stiff structure; the bracing is any of the ordinary systems, and consists of angles and tees riveted to the main girders by gusset-plates.

"TYPE 2.—Two central girders and two outside girders form its distinguishing features. This type can always be adopted in bridges up to 36′ 0″ span, and where the depth of the central girder is such that the vertical distance from the top table of the rail to the highest plane of the top range is under 2′ 6″, and the horizontal clear distance from the rail to the flange is not under 2′ 2″. This is applicable to districts where the superstructure is liable to subsidence, on account of underground workings, and also for the purposes of renewals, and future widenings or repairs, and needs no further comment. The cross-girders are spaced 7′ 4″ centre to centre, being a little less than the distance between the driving and trailing wheels of the heaviest engine; and continuous supports or rail-bearers are placed directly under the rail at 5′ 0″ centres. One disadvantage of this system, however, is that the weight of this superstructure is a little extravagant in material.

"TYPE 3.—Is in reality type 2 for larger spans up to 60′ 0″. The greater depth of the main girders causing wider flanges for lateral stiffness necessitated the 6′ 0″ being widened to 8′ 0″. It will be noticed that the cover to the fender-plates is in this type riveted on. This, however, should be removable where there is not sufficient room to get at the main girders from underneath for painting, &c. The advantage of the fender-plate is, that it keeps the web of the main girder free from the ballast, thus protecting it from moisture and lessening the dead weight.

"TYPE 4.—Two outside girders and one centre girder may also be adopted in types 2 and 3. This type 4 is the most general form of cross-section of superstructures. The weight per square foot of superstructure is less than the preceding types 2 and 3. In spans over 60′ 0″, where the clearance of 2′ 2″ has to be increased to 4′ 6″ on account of the greater depth of girder, the central girder type is at a great disadvantage. In order to conform with the Board of Trade regulations, this girder would have to be made shallow, and therefore uneconomic, necessitating an enormous number of plates—as many, perhaps, as twelve or more being needed, according to the span required. The hit-and-miss cross-girder system has the advantage of distributing the load upon the main girders more uniformly, and is also advantageous for future widenings and renewals. Where the 6′ 0″ cannot be widened, as above, to allow for the outer bridge, the cross-girders are made to span the whole width of the bridge.

"TYPES 5 & 6.—These apply where the depth of the construction has to be reduced to under 2′ 0″. Type 1 takes the form of 5, the main rail-girders being converted into trough rail-girders. This system is reverted to in all cases of limited headway up to 24′ 0″ span. The longitudinal waybeams are sunk into troughs; and the disadvantage of a fixed-sleeper road is here apparent. A layer of sand and asphalte may be laid to bed the longitudinal upon, and is an advantage, as it reduces the vibrations of the superstructure. Sometimes the waybeam is fixed direct to bent angles fixed inside the troughs. In spans of over 24′ 0″, the longitudinal trough-girders would become too deep for the depth of construction.

"TYPE 6.—Is adopted for the larger span, necessitating small depth of construction and sections. Types 2, 3, and 4 can be here resorted to with advantage. The rail-girders take here the form of trough-girders, fixed as before between the cross-girders in which the sleepers are sunk—the latter being cut in lengths to suit the distance between the cross-girders. A drawing is given, showing the details of connection of rail and cross-girders to the main girders (figs. 1′, 2′, 3′, and 4′). The width of superstructure generally adopted for single running loads is 15′ 0″ between parapets; for types 1 to 6, 26′ 0″ between parapets for two running loads. The depth of construction—that is, from top plane of the rail-table to the underside of main girders in types 2 to 4—is at least as follows:—Depth of cross-girder, 11″ to 14″; depth of broken stone and asphalte, and ballast under sleeper, 7″; from underside of sleeper to table of rail, 12½″: total, 2′ 6½″ to 2′ 9½″. For smaller depths of construction, the cross-sleepers have either to be sunk in some form of troughs, as shown in types 5 and 6, or a longitudinal waybeam may be adopted."

Floors.—The floors of any superstructure may be defined as the cross-girders or floor-beams, and rail-bearers or stringers, ties, and their necessary appurtenances. The covering to the floor is termed the decking or platform. The floors of the existing railway-bridge may be divided into five classes:—(I.) Timber-floors; (II.) cross-girders at small distances apart, varying from 2′ 0″ to 4′ 0″; (III.) cross-girders, at distances apart about 7′ 4″—a little under the distance between the driving and trailing wheels of the heaviest engines, necessitating rail-bearers; (IV.) corrugated or other forms of troughing; (V.) supports directly under the rails necessitating floor-plates only, as in type 1.

Class I.—This is the oldest type of bridge-floor. The objections to it are many: it is costly in maintenance, difficult to keep watertight, and requires protection from fire. They are now becoming obsolete.

Class II.—Cross-girders, spaced at distances varying from 2′ 0″ to 4′ 0″. In calculating the strength of the cross-girders, the dead and live loads were assumed to be uniformly distributed over the wheel-base of the engine; and thus each cross-girder was designed to carry one-quarter to one-half of the total load depending upon the spacing of the cross-girders. This assumption was not justified in practice; and the Board of Trade now requires that each cross-girder must be calculated to take the whole weight of the driving-wheels, which makes the principle most uneconomical.

Class III.—In modern bridge-floors, each cross-girder is designed to carry the whole weight on a pair of driving wheels—as much as 22 tons, concentrated at two points: this leads to spacing them at distances apart not less than that between the driving and trailing wheels of the heaviest engine. This gives an economical distribution of the material, but necessitates additional girders, called rail-bearers, to carry the load between the cross-girders; they also serve as bracing between the cross-girders; and floor-plates of any approved type are riveted to these, making a watertight floor. Three inches of broken-stone asphalte is laid upon the floor-plates, with at least four inches of ballast under the sleepers.

Class IV.—Trough-floorings of recent years have come much into favour. The troughs are laid transversely, or in small spans parallel with the rail. In this system of flooring, simplicity of construction is obtained. The load is more uniformly distributed along the main girders. They are also solid floors with or even without ballast; they present a continuous surface of uniform strength against possible derailment. They can also give a continuous floor-surface, take up little depth, form an excellent lateral bracing, and are extremely light.

Many types of these trough-floors are now on the market, inventors having turned their attention in this direction, with the result that there is now a plethora of types. Three of these, which combine the distinctive features of the majority, and which are in common use, are Lindsay's, Hobson's, and Westwood's, shown respectively on figs. 58, 59, 60. By permission of Messrs. Dorman, Long, and Co., and the Motherwell Iron and Bridge Co., particulars of the usual market sizes of these troughs are given in the accompanying tables, together with the weights in pounds per square foot, the moments of inertia measured in inch-units, and the moments of resistance measured in inch-tons.

[FIGS. 58, 59, 60, *and* TABLES *follow here.*

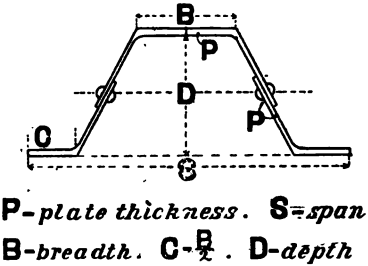


Fig. 58.—Lindsay's Troughs. For Table, see page 87.

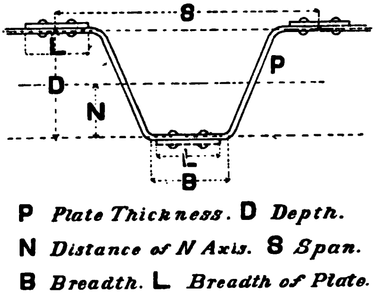

S
L
P
D
N
L
B
P Plate Thickness. D Depth.
N Distance of N Axis. S Span.
B Breadth. L Breadth of Plate.

Fig. 59.—Westwood's Troughs. Fo Table, page 88.

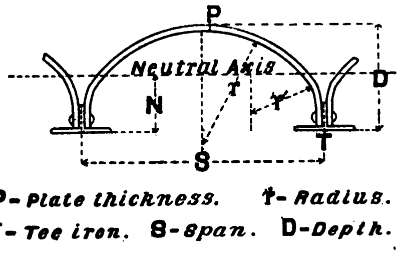


Fig. 60.—Hobson's Arched Troughs. For Table, see page 89.

LINDSAY'S TROUGHS.

Sect.	S.		D.		B.	P.	P'.	Weight per square foot.	Moment of Resistance.
	ft.	in.	ft.	in.	in.	in.	in.		
A	1	0	0	$4\frac{1}{4}$	4	$\frac{3}{8}$	$\frac{3}{16}$	19·06	42·57
B	1	0	0	4	4	$\frac{1}{4}$	$\frac{1}{8}$	13·40	32·00
C	1	2	0	$5\frac{1}{4}$	$4\frac{1}{2}$	$\frac{7}{16}$	$\frac{1}{4}$	23·10	71·82
D	1	2	0	5	$4\frac{1}{2}$	$\frac{5}{16}$	$\frac{3}{16}$	17·61	54·60
E	1	4	0	$6\frac{1}{4}$	$5\frac{1}{8}$	$\frac{1}{2}$	$\frac{5}{16}$	28·00	113·75
F	1	4	0	6	$5\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	21·80	87·75
G	1	8	0	$7\frac{3}{8}$	6	$\frac{5}{8}$	$\frac{3}{8}$	32·97	198·90
H	1	8	0	7	6	$\frac{7}{16}$	$\frac{9}{32}$	24·52	140·53
I	2	0	0	$9\frac{3}{8}$	$7\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{8}$	35·02	334·42
J	2	0	0	9	$7\frac{1}{2}$	$\frac{7}{16}$	$\frac{9}{32}$	26·26	237·70
K	2	8	1	$0\frac{1}{2}$	9	$\frac{3}{4}$	$\frac{1}{2}$	40·50	673·00
L	2	8	1	1	9	1	$\frac{5}{8}$	51·83	882·70
M	2	8	1	0	9	$\frac{1}{2}$	$\frac{3}{8}$	28·78	472·35
N	3	0	1	$5\frac{1}{4}$	9	1	$\frac{5}{8}$	56·76	1325·15
O	3	0	1	$3\frac{3}{4}$	9	$\frac{1}{2}$	$\frac{3}{8}$	32·18	677·50

G, suitable for single line of railway.

I, suitable for carrying railway over small spans.

N, suitable for double line of railway.

Sections N and O are built up of sections L and M, a strip of plate forming a lap-joint connecting the troughs together.

In calculating the moment of resistance, a working-stress of 6·5 tons per square inch has been used.

WESTWOOD'S TROUGHS.

Table	S.		D.	N.	L.	B.	P.	Weight per sq. ft.	Moment of Inertia.	Moment of Resistance.
	ft.	in.	in.	in.	in.	in.	in.			
1	2	0	7½	3·75	6	7⅛	½	42·40	286·72	1735·70
	2	6	10	5	8	9⅛	½	43·40	650·70	3101·20
	2	8	12	6	8	9¼	½	43·50	947·50	3858·90
	2	10	14	7	8	9½	½	45·30	1339·20	4761·70
	3	0	16	8	8	9¾	½	45·80	1788·00	5641·20
2	1	8	6	3·50	5	—	½	34·50	109·05	806·20
	2	0	7½	4·50	6	7⅛	½	33·80	193·00	1137·68
	2	6	10	5·87	8	9⅛	½	35·02	442·42	2022·49
	2	8	12	7	8	9¼	½	35·50	660·70	2551·60
	2	10	14	8	8	9½	½	37·80	962·00	3265·00
	3	0	16	9	8	9¾	½	38·70	1308·36	3960·40
3	1	8	6	3·42	3	—	½	32·90	104·54	799·50
	2	0	7½	4·25	3	7⅛	½	31·20	180·72	1124·48
	2	6	10	5·50	3	9⅛	½	31·55	400·32	1949·38
	2	8	12	6·50	3	9¼	½	32·49	598·14	2481·17
	2	10	14	7·50	3	9½	½	35·00	879·35	3177·00
	3	0	16	8·50	3	9¾	½	36·00	1198·60	3835·70

The sections in table 1 have straps on top and bottom of trough, the straps being ⅛″ thicker than the trough.

In the sections of table 2 the bottom strap is omitted, the top strap being same thickness as trough.

In the section of table 3 the troughs are lap-jointed.

Nearly all the sections are supplied in ⅜″, ½″, and ⅝″ thicknesses of trough. The weights per square foot, moments of inertia, and moments of resistance for any other section are closely proportional to the thickness of the trough: hence, the above numbers multiplied by ·75 for ⅜″, and by 1·25 for ⅝″ troughs will give the required quantities.

HOBSON'S ARCHED TROUGHS.

S.	D.	N.	r.	r′.	T.	P.	Weight per square foot.	Moment of Inertia.	Moment of Resistance.
ft. in. 1 9	in. 9	in. 4·91	in. 10½	in. 4¾	in. in. in. 5 × 2½ × ⅜	in. ¼	20·31	105·83	581·97
2 0	10	5·76	36	6½	5 × 3 × ⅜	¼	20·60	146·76	689·33
1 10½	9½	5·17	28½	5⅜	7 × 3½ × ½	⅜	33·14	218·82	1154·84
2 3	12	6·54	13	7½	6 × 3 × ½	5/16	25·67	309·64	1278·35
2 6	12	6·97	15¾	9	5 × 3 × ⅞	¼	19·78	266·37	1031·42
2 6	12	6·61	36	8¼	6 × 3 × 7/16	5/16	23·95	321·79	1313·86
2 6	12	6·45	36	8¼	6 × 3 × ⅝	⅜	29·53	400·59	1675·86
2 6	15	8·39	FLAT.	12	4½ × 3½ × 7/16	5/16	24·48	492·93	1584·78
2 6	15	8·30	36	11½	6 × 3 × ½	5/16	26·38	545·71	1774·40
2 6	15	8·72	36	11½	6 × 4 × 11/16	⅜	33·00	682·89	2270·40

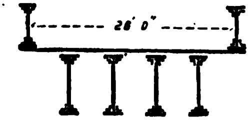
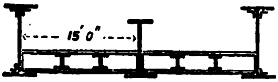
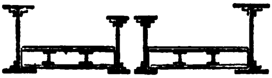
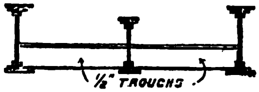
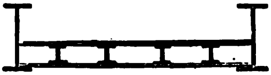
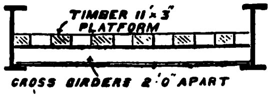
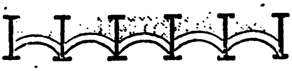
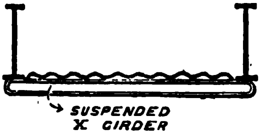
From § 31 the moment of resistance $= \frac{pI}{y} = M$. The value of $p = f$; the working strength of the material used in calculating the moment of resistance is $6\frac{1}{2}$ tons per square inch for Lindsay's troughs. For Hobson's and Westwood's troughs the value of f used is the breaking-strength of the steel, and has been taken at 27 tons per square inch, the factor of safety required being left to the discretion of the designer.

To use the tables:—To determine the section required for a given loading and span, calculate the maximum value of the moment of the external forces, and select that section whose moment of resistance is nearest to it. To determine the span for a given uniformly distributed load. From § 20 $M = \frac{WL}{8}$: hence, $L = \frac{8}{W} \times$ moment of resistance. Also the load for a given span is $W = \frac{8}{L} \times$ moment of resistance.

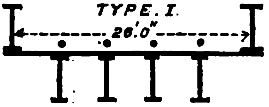
The ratio of the depth to span should not exceed $\frac{1}{20}$ to provide against excessive deflection; that is, so as to ensure sufficient stiffness (§ 34).

A comparison of the weights of bridge-floors referred to now follows, together with tables giving the actual weights of underbridges of the types referred to in § 64.

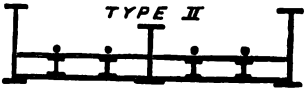
Comparison of the Weight of Bridge Floors.

Type.	Weight in lbs. per sq. ft.	Remarks.
Underbridges.—		
	28	⅜in. floor plates
	54	Cross girders, rail bearers, and ⅜in. floor plates.
	54	Do. do. do.
	43	½in. Westwood and Baillie's trough rolled plates, 12in. deep
	68	Cross girders spanning whole width of 26ft.
	39	Cross girders with timber planking
Overbridges.—		
	264	Longitudinal wrought iron girders. Jack arches and masonry (including backing)
	78	Main girders, cross girders, and corrugated plates.

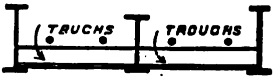
Actual Weights of Underbridges.



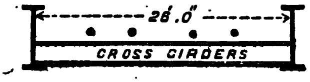
Skew span.	Total weight of parapet girders.	Total weight of girder under rail.	Total weight of flooring.	Total weight of bridge.	Weight of flooring per sq. ft. in lb.	Weight of bridge per sq. ft. in lb.
ft. in.	t. c. q. lb.	t. c. q. lb.	t. c. q. lb.	t. c. q. lb.	lb.	lb.
23 6	2 4 2 5	2 3 2 1	8 9 3 2	21 12 3 7	31·12	79·83
25 0	2 5 2 18	2 5 1 14	7 11 3 1	22 9 1 27	26·15	77·45
25 0	2 4 1 15	2 5 1 14	7 9 1 0	22 1 2 0	25·72	76·07
26 0	2 6 2 18	2 11 1 16	7 11 2 20	22 12 1 4	25·28	74·93
36 0	3 4 1 3	4 8 2 19	11 8 3 14	35 7 2 17	27·39	84·67
36 3	3 4 1 3	4 8 2 26	11 9 0 27	35 18 2 22	27·24	84·76
58 6	5 7 0 7	9 10 0 24	19 19 2 16	67 4 1 6	29·48	98·98
61 10	8 7 1 3	10 15 0 0	20 3 1 23	76 14 0 12	28·17	106·80
78 4	7 16 1 7	16 11 3 10	20 15 3 19	103 16 3 5	22·87	114·18



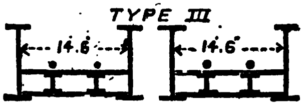
Skew span.	Weight of outside girder (each).	Weight of inside girder (each).	Total weight of flooring.	Total weight of bridge, exclusive of parapet.	Weight of flooring per sq. ft. in lb.	Weight of bridge per sq. ft. in lb.
ft. in.	t. c. q. lb.	t. c. q. lb	t. c q. lb	t. c. q. lb.	lb.	lb.
24 0	*9 16 0 0		16 7 0 0	26 3 0 0	57·58	92·00
30 0	3 1 1 11	4 16 2 25	16 15 1 12	27 14 2 21	48·14	79·65
30 0	3 13 1 10	5 0 0 11	17 4 0 18	29 10 3 24	49·42	84·85
37 6	*20 14 2 0		22 6 0 0	43 0 2 0	49·90	96·87
45 0	7 2 2 0	10 9 2 0	26 7 0 6	51 1 2 6	50·46	97·79
50 0	*36 12 0 0		30 15 3 0	67 7 3 0	52·05	113·91
51 7	*40 1 0 0		35 1 0 0	75 2 0 0	56·92	122·08
62 0	*48 19 0 0		41 1 0 0	90 0 1 0	55·42	121·59
68 0	16 11 2 0 15 10 3 6	25 3 0 0	52 16 0 0	107 0 0 0	56·80	135·25
25 7	*12 4 0 0		12 5 0 0	24 9 0 0	41·05	82·28
30 8	*18 12 2 7		19 12 1 0	33 4 3 7	43·51	90·69
79 8	16 8 3 14	30 0 1 21	52 7 2 21	115 5 3 14	45·91	101·04



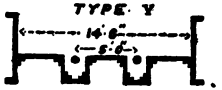
* Weights of three main girders (inclusive) in tons.



Skew span.	Weight of outside girders (each).	Total weight of flooring.	Total weight of bridge.	Weight of flooring per sq. ft. in lb.	Weight of bridge per sq. ft. in lb.
ft in.	t. c. q. lb.	t. c q. lb.	t. c. q. lb.	lb.	lb.
88 0	8 9 1 4	83 15 1 20	51 14 0 0	76·56	117·21
61 1	16 10 2 22	48 8 2 19	81 10 0 7	68·82	114·96
68 2	—	—	90 10 2 11	—	128·50
78 0	29 0 8 0	54 8 1 0	112 4 8 1	60·00	128·97
80 7	89 2 0 0	69 8 0 0	147 7 0 0	72·51	154·52
80 8	—	—	126 2 0 0	—	184·68
108 6	—	—	216 4 1 8	—	171·68
120 0	46 12 8 12	82 0 0 0	175 0 0 0	—	—
132 6	97 10 0 0	96 7 0 8	291 7 0 8	62·65	189·44



61 9	17 2 2 0	18 6 1 12	53 17 0 2	48·14	70·02



	Parapet and trough girdgers.	Flooring plates.	Total weight of bridge.		
*12 0	3 17 1 11	1 16 1 10	5 14 0 7		74·06
†14 3			10 14 0 14		69·08

* Single road. † Double road.

THE WEIGHT OF PLATE-GIRDERS.

65. The problems involved in determining the weights of plate, lattice, or triangulated girders, have hitherto proved so complex that, so far as the author is aware, no general formulæ have been given which admit of general use in practice.

In what follows, an analysis is given which admits of easy application in determining these quantities, and from which some interesting deductions are given, comparing the relative economy of different types of girders.

The application of the analysis to the case of a plate-girder under concentrated and distributed loads will be first treated of, other types of girders and distributions of load being afterwards considered.

The fundamental relations that obtain between the load, shearing-force, and bending-moment diagrams, when applied to the case of a girder resting on two supports, have already been treated of in §§ 21 and 25. Let a system of loads be placed on the span of a girder; then the shearing force at any section of the span is the algebraic sum of the forces—that is, of the loads and reactions on either side of the section; and by determining these values for a number of sections, the shearing-force diagram is obtained.

Now, all systems of loads are either concentrated or distributed. Let W_1, W_2, . . . W_n refer to the concentrated loads, and w_1, w_2, . . . w_n to the distributed loads on the span; then the shearing force at any section is

$$F = \Sigma W + \int w dx;$$

or, expressed in words, the ordinates of the shearing-force diagram are proportional to the algebraic summation of the concentrated loads and one reaction, together with a load proportional to the area of the diagram of the distributed loads to either side of the section.

The ordinates of the bending-moment diagram are determined by integration from the shearing-force diagram; that is,

$$M = \int F dx = \Sigma W x + \int\int w dx^2.$$

From this equation, it follows that the bending moment at any section is proportional to the area of the shearing-force diagram measured from the ends of the span to that section. The bending moment will have its maximum value where the shearing-force diagram cuts the base, the areas to either side of the point then being equal to one another; that is, the base line of the shearing-force diagram divides the diagram into two equal areas. For girders simply supported at the end of the span, the relation is always true under all conditions of loading.

With rectangular girders, the following relations will be shown to be true:—*That the theoretical volume of the web in plate-girders, and of the struts and ties in lattice and triangulated girders, is proportional to the area of the shearing-force diagram, and independent of the depth of the girder.* This remarkable fact seems to have been first pointed out by Mr. Stoney, and greatly simplifies the analysis for the volume of such girders. *The volume of the flanges is proportional to the area of the bending-moment diagram, and varies inversely as the depth.*

In the analysis for the plate-girder, the web will be assumed to be strained along lines inclined at 45° to the vertical. From theoretical considerations, it has been shown that the lines of stress in a beam cut the neutral axis at 45°, and that for practical purposes this angle gives the nearest approximation to the direction of the lines of stress in the web (§ 44).

It is convenient to consider the web as being made up of two parts, one in tension and one in compression; this assumption avoids the difficulties that would occur if the web were treated as being subjected to compound stress (§ 45), and at the same time puts the analysis in the same form as will be required for lattice and triangulated girders.

Now, consider a simple stress of intensity f acting through one of the bars; then, on the vertical cut, the shear stress would be $f \sin \theta \cos \theta$; and since $\theta = 45°$, the shear stress $s = f/2$; also the normal stress would be $f \cos^2 \theta$, giving again $f/2$ when $\theta = 45°$.

66. Rectangular Girder 45° Bracing. Concentrated Load.

—On fig. 61 a rectangular girder—that is, a girder with parallel flanges, with 45° bracing and vertical end-struts—is shown carrying a concentrated load in the centre of the span. It is required to determine the theoretical volume of the girder.

Volume of Flanges.—The bracing is inclined at 45°, and the shear stress assumed to be uniformly distributed over the web. Under these conditions the bending moments are wholly transmitted by the flanges.

The bending moment at the centre is $M = \frac{WL}{4}$; this is equal to *fad*, where f = the working strength of the material, a = the area of the cross-section, and d = the depth of the girder. Hence, $a = \frac{WL}{4fd}$. The bending moments at the ends are nothing, and, at any intermediate point, vary directly as its distance from the ends of the span. Therefore the average area of the cross-section of the flange will be one-half the maximum $= \frac{WL}{8fd}$; the volume for the two flanges will be the area of the cross-section × 2-length: hence, $V_0 = \frac{WL^2}{4fd}$. Let $\frac{d}{L} = r$, the ratio of the depth to span; then $V_0 = \frac{WL}{4fr}$.

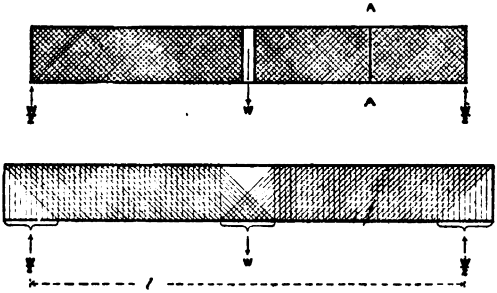


Fig. 61.

Volume of Web.—The shear stress over any vertical cross-section is $\frac{W}{2}$; this equals $a_1 f/2$ where a_1 is the area of the cross-section of the web, and $f/2$ the intensity of the shear stress: hence, $a_1 = \frac{W}{f}$, and the volume is $a_1 L = V_1 = \frac{WL}{f}$; that is, the volume of the web does not depend on the depth of the girder.

The relation that the theoretical volume of the web is independent of the depth of the girder is true within wide limits. It is true for all conditions of loading, and applies equally to triangulated or lattice-girders.

Volume of End-Struts and Vertical Tie.—To distribute the load over the lattice-bracing, a vertical tie shown in the centre of the span is required. The area of its cross-section at the lower flange would be $\frac{W}{f}$, and at the top flange would be nothing, giving an average of $\frac{W}{2f}$. The volume is the area × depth, equals

$$\frac{Wd}{2f} = \frac{WL}{2f}r.$$

The two end-struts carry each a load of $\frac{W}{2}$, the area of the cross-section on the lower flange being $\frac{W}{2f}$, and nothing at the top. Giving a mean area of cross-section of $\frac{W}{4f}$, the volume for the two struts will be

$$\frac{W}{4f} \times 2d = \frac{Wd}{2f} = \frac{WL}{2f}r.$$

The volume for end-struts and vertical tie then is

$$V_2 = \frac{WL}{f}r.$$

Depth for Minimum Volume.—The total volume of the girder is

$$V = V_0 + V_1 + V_2 = \frac{WL}{f}\left(\frac{1}{4r} + 1 + r\right)$$

Differentiating the expression with respect to r, the equation is

$$\frac{dV}{dr} = \frac{WL}{f}\left(-\frac{1}{4r^2} + 1\right) = 0$$

for the minimum value, giving $r = \frac{1}{2}$.

Substituting this value, the expression becomes

$$\frac{WL}{f}\left(\tfrac{1}{2} + 1 + \tfrac{1}{2}\right),$$

which shows that the volume of the flanges, end-struts, and vertical tie is equal to the web, also that the total volume

$$V = \frac{2WL}{f}.$$

From the application of the analysis to triangulated and lattice-girders (§ 71), it will be seen that, for the minimum volume, the volume of the flanges, together with the volume of the resolved part of the web transmitting the horizontal stresses, is equal to the volume of the end-struts, vertical ties, and the volume of the other part of the web transmitting the vertical stresses, and that this relation applies generally and to various types of loading.

For *rectangular girders*—that is, girders in which the internal system of triangulation admits of being indefinitely reduplicated, with the triangulation at 45°—the depth with the concentrated load that gives the minimum volume is invariably one-half the span. The result, however, is only of theoretical interest.

Stiffening of Girders.—What is it, then, that fixes the economic depth of the girder? It is the material required to stiffen the web and flanges to enable them to transmit direct thrust, the allowance required for connecting the flanges together to transmit twisting moments, owing to the loads not being applied in the vertical plane of the girder, and the material required to allow for wind-pressure.

It is impossible to neglect these quantities in any practical analysis on the volumes of girders; and, as will appear immediately, it is possible to introduce a coefficient which will allow for them with a fair degree of accuracy.

Rectangular Girder with 45° Ties and Vertical Struts. Concentrated Load.—If, instead of the bracing being at 45°, the ties are supposed to be at 45°, and the struts vertical, as shown on fig. 61, what then will be the volume of the girder?

Volume of Flanges.—The volume of these will be, as before,

$$V_0 = \frac{WL}{4fr}.$$

Volume of the Web.—The volume of the ties will be as before $= \frac{WL}{f}$. In addition to this there is the volume of the vertical struts. The vertical component of the stress in inclined ties per unit of length along the flanges is $\frac{W}{2d}$; and the area of the cross-section of the strut $= \frac{W}{2fd}$. The volume is the area of the cross-section of the strut × $Ld = \frac{WL}{2f}$. The total volume

$$V_1 = \frac{WL}{f}(1 + \tfrac{1}{2}) = \frac{3}{2}\,\frac{WL}{f}.$$

This includes the volume for the end-struts in this case. The expression for the total volume is

$$V = V_0 + V_1 = \frac{WL}{f}\left(\frac{1}{4r} + \frac{3}{2}\right).$$

As it stands, it does not give a finite value of r for the minimum value. A more rigorous analysis shows that the minimum again occurs when $r = \tfrac{1}{2}$.

67. **Rectangular Girder 45° Bracing. Uniform Load.**—*Volume of Flanges.*—The volume of the plate-girder shown on fig. 62 will now be determined, the load on the span being uniformly distributed; the shearing-force and bending-moment diagrams are given on figs. 9, 10.

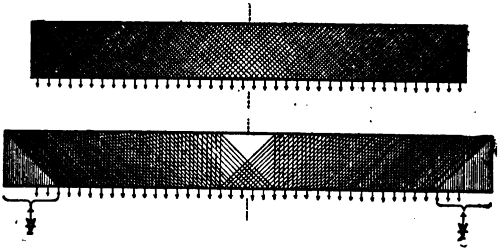


FIG. 62.

Let w = the load in tons per foot on the span, L = the length of the span, and $c = L/2$. Take the origin of the coordinates at the centre of the span: then the moment M_x at any section is $w/2\,(c^2 - x^2)$ (§ 20). Suppose the flanges to take up the whole of the bending moments: then, since the depth = d is constant, the cross-section of the flanges will vary directly as the bending moment. Thus, let f = the working stress of the material, a = the area of the cross-section of the flanges, then

$$fad = M = \frac{w}{2}(c^2 - x^2) \quad \text{and} \quad a = \frac{w}{2fd}(c^2 - x^2).$$

The volume of any small element of the flanges along the span is

$$\delta V = \frac{w}{2fd}(c^2 - x^2)\,\delta x:$$

hence the volume of the two flanges will be

$$V_0 = \frac{w}{fd}\int_{-c}^{c}(c^2 - x^2)\,dx = \frac{4}{3}\frac{wc^3}{fd} = \frac{WL^2}{6fd} = \frac{WL}{6fr},$$

where $r = \frac{d}{L}$, the ratio of the depth to span. The assumption that the flanges take up the whole of the bending moments is only true when the shear stress is supposed to be uniformly distributed over the web. A more rigid investigation given below shows that a more accurate expression for the volume is

$$V_0 = \frac{WL}{6fr}(1 - r^2) = \frac{WL}{6fr}\left(1 - \frac{1}{4n^2}\right),$$

where n = the number of turns to-and-fro of any one triangulation in the half span. To ensure no bending on the end-struts the value of n should be some whole number. For the usual ratios of r in practice, this effect is small, and may be neglected.

In the equations $\frac{1}{2n} = r$.

Suppose the web of the girder with the 45° bracing, shown on fig. 62, to be developed so as to form a 45° right-angled triangle on the half span: then the shear stress $s = w \tan\theta$, and $\tan 45° = 1$; therefore, $s = w$.

The area of the cross-section of the flange at the centre of the span to carry this shear stress would be

$$a = \frac{w}{f}\frac{l}{2};$$

and the volume v of one-half of one flange

$$= \frac{al}{2} = \frac{w}{f}\frac{l^2}{4} \times \tfrac{1}{2},$$

the coefficient $\frac{1}{2}$ being used because the average cross-section is half the maximum. Now, let the developed triangle of the web be doubled to-and-fro n times: then the summation of the volumes for one-half the upper and lower flanges would be of the form

$$\tfrac{1}{2}V_0 = \frac{w}{f}\left[\begin{array}{l}\left\{\dfrac{ln}{2n}\right\}\left\{\dfrac{l}{2n}\,\dfrac{n}{2}\right\}\\ +\,2\left\{\dfrac{l(n-1)}{2n}\right\}\left\{\dfrac{l}{2n}\,\dfrac{(n-1)}{2}+\dfrac{l.1}{2n}\right\}\\ +\,2\left\{\dfrac{l(n-2)}{2n}\right\}\left\{\dfrac{l}{2n}\,\dfrac{(n-2)}{2}+\dfrac{l.2}{2n}\right\}\\ \qquad\text{etc., to } n-1 \text{ terms.}\end{array}\right]$$

The summation of this series is

$$\tfrac{1}{2}V_0 = \frac{w}{f}\left(\frac{l}{2n}\right)^2\left[\frac{n^2}{2}+\left\{2(n-1)\left(\frac{n-1}{2}+1\right)\right\}\right.$$
$$\left.+\left\{2(n-2)\left(\frac{n-2}{2}+2\right)\right\}, \text{etc.}\right]$$
$$= \frac{w}{f}\left(\frac{l}{2n}\right)^2\left[\frac{n^2}{2}+(n^2-1^2)+(n^2-2^2)+\text{etc.}\right]$$
$$= \frac{w}{f}\left(\frac{l}{2n}\right)^2\left[\frac{2}{3}n^3-\frac{n}{6}\right]$$
$$= \frac{wl^2}{12f}\left[2n-\frac{1}{2n}\right];$$

and
$$V_0 = \frac{wl^2}{6f}\left(2n-\frac{1}{2n}\right) = \frac{Wl}{6fr}\left(1-\frac{1}{4n^2}\right)$$
$$= \frac{wl^2}{6fr}(1-r^2) = \frac{Wl}{6fr}(1-r^2).$$

Volume of the Web.—The shear stress over any cross-section is proportional to the ordinate of the shearing-force diagram (fig. 10) at that section, and equals wx where w is the load in tons per foot on the girder. At the ends of the span this becomes

$$\frac{wL}{2} = \frac{W}{2},$$

and gives the maximum value of the shear stress over the ends of the girder. The intensity of the shear stress is $f/2$: hence, the area of the cross-section at the ends

$$= \frac{W}{2}\,\frac{2}{f} = \frac{W}{f}.$$

The volume is the average area of the cross-section

$$= \frac{W}{2f} \text{ multiplied by the length } = \frac{WL}{2f};$$

that is, the volume of the web is one-half of that for the concentrated load with the 45° bracing. It will be remembered that the areas of the shearing-force diagrams for the concentrated load and the uniform load are in the proportion of two to one: hence it follows that the volumes of the web are proportional to the areas of their respective shearing-force diagrams.

Volume of End-Struts.—The end-struts transmit the shear stress from the bracing to the abutments; and the cross-section of the strut will vary from nothing at the top to $\frac{W}{2f}$ at the bottom, giving a mean cross-section of $\frac{W}{4f}$: the volume of the strut will be

$$\frac{Wd}{4f} = \frac{WL}{4f}r;$$

and for two struts the volume is

$$V_2 = \frac{WL}{2f}r.$$

The vertical ties in this case are neglected, as the load can generally be transmitted to the bracing from the lower flange; and the ties are therefore not required. For the case where the depth is half the span, one-half of the load in the central part of the span must be supposed to be distributed on the central ties; and the case reduces to that of the concentrated load, thus:—

Total Volume.—$V = V_0 + V_1 + V_2$

$$= \frac{WL}{f}\left(\frac{1}{6r} - \frac{r}{6} + \frac{1}{2} + \frac{r}{2}\right).$$

The value of n cannot be < 1, which gives $r = \frac{1}{2}$ for the minimum volume. A more rigorous analysis—given in § 72—which allows for the vertical ties, gives the value of r slightly under $\frac{1}{2}$. As before mentioned, these cases are only of theoretical interest. For values of $r \leqq \frac{1}{8}$, no difficulty arises with regard to the distribution of the load from the flanges, and no vertical ties are required.

Rectangular Girder—Vertical Strut and Ties at 45°. Uniform Load.—On fig. 62, a diagrammatic sketch of the girder is shown.

Volume of Flanges.—These will be the same as for the 45° bracing, and equal to

$$V_0 = \frac{WL}{6fr}(1 - r^2) = \frac{WL}{6fr}\left(1 - \frac{1}{n^2}\right).$$

It will be noted that n has twice the value of that for the 45° bracing.

Volume of Web and Struts.—For the inclined ties, this is the same as that for the 45° bracing $= \frac{WL}{2f}$; for the vertical struts, resolve the stresses in the inclined ties similarly, as in the case for the concentrated load: the value then obtained for the volume is $\frac{WL}{4f}$, giving a total volume $V_2 = \frac{3}{4}\frac{WL}{f}$. As in the case of the concentrated load, the volume of the end-struts is included in this quantity.

68. **Live and Dead Loads.**—In addition to the dead load on a girder, it is necessary to take into account the live or moving load.

The shearing-force diagram for three loads was given on § 25; also, the statical loads that would give this shearing-force diagram (§ 26) are:—

A uniform dead load of $W + \frac{U}{2}$, a concentrated load of $\frac{U}{4}$ placed at the centre of the span, and a pair of wedge-shaped loads, each of $\frac{U}{8}$, varying from nothing at the centre to a maximum at the ends of the span.

Now, consider the application of these diagrams in determining the volume of the web for a plate-girder, remembering that the volume of the web is proportional to the area of the shearing-force diagram; the volumes for the concentrated, uniform, and wedge-shaped loads are in the proportion of 1, $\frac{1}{2}$, $\frac{1}{3}$. This has been shown for the concentrated and uniform loads; and a similar demonstration applies for the

wedge-load. The following table gives these volumes in terms of the loads given on the load-diagram:—

$$\text{Volume of web} = \frac{L}{f} \times$$

Loads.	Ties and Struts inclined at 45°.	Vertical Struts and Ties inclined 45°.
Concentrated .	$\frac{U}{4}\left(1+\frac{r}{2}\right)^*$	$\frac{3}{8}U$
Uniform . .	$\frac{1}{2}\left(W+\frac{U}{2}\right)\left(1+r\right)$	$\frac{3}{4}W+\frac{3}{8}U$
Wedge-shaped .	$\frac{1}{3}\frac{U}{4}(1+\frac{3}{2}r)$	$\frac{1}{8}U$
Total volume .	$=\frac{L}{f}\times\left[\left(\frac{W}{2}+\frac{7}{12}U\right)+\frac{1}{2}(W+U)r\right]$	$\frac{L}{f}\times(\frac{3}{4}W+\frac{7}{8}U)$

The volume of the web is determined in terms of these coefficients for the live and dead loads. The value of $\frac{r}{2}^*$ has been taken, instead of r, for the concentrated load; because no vertical tie is required, and only the end-struts require to be considered.

Volume required for Stiffening the Girder.—If the theoretical volumes of the flanges and web were alone to be considered, the economic depth would be given when $r = \frac{1}{2}$, nearly; this will be evident from the formulæ just given.

In determining the practical economic depth, it is necessary to allow for the volume of material for stiffening the web as a strut; for stiffening the connections of the flanges together by gusset-plates, to take up the twisting moments caused by the loads not acting in the vertical plane of the girder; for stiffening the gusset-plates and struts to transmit wind-pressure; and lastly, to allow for stiffening of the flanges considered as struts.

The allowance for these quantities will be made in two ways: firstly, by adding to the volume of the web and flanges a fixed percentage of their volume; and secondly, by adding a variable quantity depending on the square of the depth.

Plate-girders, in practice, are usually built with vertical struts, and approximate more nearly to the type of girder with vertical struts and inclined ties than to that with the 45° bracing. Using the coefficients obtained for this type, the volume of the web is

$$\frac{L}{f}\left(\tfrac{3}{4}W + \tfrac{7}{8}U\right).$$

Now, to allow for butt-straps, joints of connections, the extra material (owing to the rivets reducing the efficiency of joints), and for the gusset-plates, about 25 per cent. of the theoretical volume will be added to the web—that is,

$$\frac{L}{f}\left(\tfrac{1}{4}W + \tfrac{1}{8}U\right),$$

giving for the total volume of the web

$$V_1 = \frac{(W + U)L}{f}.$$

For the stiffening of the web as a strut, the Rankine-Gordon formula (§ 51) will be used in the following form:—

$A_1 = A_0 c \left(\frac{l}{\rho}\right)^2$ where A_1 is the additional area of the cross-section to resist bending, A_0 the area of a short strut to carry the given load, $c = \frac{1}{9000}$ the coefficient for round-ended struts, ρ = the radius of gyration of the section, and l = the length of the strut.

Now, consider the plate-girder with vertical struts and inclined ties, the theoretical volume of the portion of the web which acts as a strut under a uniform load in this type is $\frac{WL}{4f}$; this is constant and independent of the depth. If A_0 is the cross-section of the struts, and d the depth of the girder, then $A_0 d = \frac{WL}{4f}$: but $l = d = Lr$; therefore $A_0 = \frac{W}{4fr}$. Substituting in the Rankine-Gordon formula

$$A_1 = \frac{Wc}{4fr}\left(\frac{Lr}{\rho}\right)^2,$$

the volume for stiffening the struts is

$$A_1 d = A_1 Lr = \frac{WLc}{4f}\left(\frac{Lr}{\rho}\right)^2.$$

If V_2 is the volume for stiffening, and V_1 the theoretical volume for the struts $= \frac{WL}{4f}$, then

$$V_2 = V_1 c\left(\frac{Lr}{\rho}\right)^2.$$

In a practical girder, with $r = \frac{1}{10}$, the value of $\frac{d}{\rho} = \frac{Lr}{\rho}$ was from 65 to 70—that is, the mean breadth of the struts was about $\frac{1}{20}$ the depth; substituting these values $c\left(\frac{Lr}{\rho}\right)^2 = \cdot 5$, for any other depth the volume will be

$$V_2 = V_1 \times \cdot 5 \times 100r^2$$

$$= \frac{WL}{4f} \times 50r^2;$$

and for the live and dead loads the volume is

$$\frac{W + U}{4} \cdot \frac{L}{f} \cdot 50r^2:$$

this gives the required volume, the value of ρ being assumed constant. With the bracing at 45°, a similar investigation leads to an analogous result.

It is to be remembered that if the web were stiffened with this volume, so as to enable it to transmit direct thrust, yet the girder as a whole would be unstable; this will be evident by assuming the struts to have round ends. Hence, in addition to the above volume, there is still to be considered the volume required to stiffen the girder against wind-pressure, and to withstand the twisting of the flanges.

The allowance for these can be only made empirically. In a particular girder of $\frac{1}{10}$ the span, the volume of material for stiffening struts, gusset-plates, &c., was sensibly ·5 the volume of the web. Now, for the struts, the volume required for stiffening varies as r^2; also for wind-pressure, the volume varies as r^2; and for stiffening against distortion by twisting, the volume varies as r^2. Hence, for the particular girder of $\frac{1}{10}$ the span, the sum of these volumes added together would be $(W + U)\frac{L}{f} \times \cdot 5$; and for any other girder, *the sum of the volumes for struts, wind-pressure, and twisting of the flanges* could be put in the form

$$V_2 = (W + U)\frac{L}{f}(\cdot 5 \times 100)r^2.$$

Volume of Flanges.—The stresses in flanges will be a maximum when the live and dead loads are on the whole of the span; and the volume of the flanges for the 45° bracing and for the vertical struts and inclined ties will be

$$V_0 = \frac{(W + U)L}{6fr}(1 - r^2).$$

The practical volume is, however, greater than this, provision having to be made to enable the flanges to transmit thrust as a strut; also allowance must be made for the reduced section of the flanges, owing to rivets. Assume the value of $\frac{L}{\rho}$ to be constant, and not to depend on the depth of the girder, then the additional material required to stiffen the flanges as a strut will be a certain percentage of the volume of the flanges, and this will also be true for the rivets.

Further, the term $- r^2$ will also be neglected, as in all practical cases its value is too small to be of any consequence, and the increase of volume given by neglecting it would require to be allowed in other ways if it was retained.

To stiffen the flanges, at least 20 per cent. requires to be added to the theoretical volume; if this is done, the volume takes the form $V_0 = \frac{(W + U)}{5}\frac{L}{fr}$: this value will be adopted.

69. **Formula for the Weight of Girders.**—Collecting the terms for the volumes of the flanges, web, and for stiffening, the following formula is obtained:—

$$V = V_0 + V_1 + V_2 = (W + U)\frac{L}{f}\left(\frac{1}{5r} + 1 + 50r^2\right),$$

W and U being the dead and live loads on the span in tons, L the length in inches, and f the working-strength of the material in tons per square inch.

This formula was obtained on the assumption that the ties were inclined at 45°, and the struts vertical; if, however, the bracing had been at 45°, the formula would have taken the form of

$$(W + U)\frac{L}{f}\left(\frac{1}{5r} + A + Br + Cr^2\right).$$

A, B, and C being constants, a formula in which these constants were determined was obtained, giving results which differed only in a small degree from the one already given.

If this formula is tabulated for various values of r, it will be found that, within a wide range of the depth, the volume of the girder is sensibly constant, and that the minimum volume occurs when $r = \frac{1}{8}$, as will be seen at once by differentiating the above expression. For values of $r = \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$, the constant is sensibly 3·5. Adopting this value, the formula takes the simplified form

$$V = \frac{(W + U)L \times 3{\cdot}5}{f};$$

and expressed in tons, the weight of the girder is

$$W_1 = \frac{(W + U)L \times 3{\cdot}5}{8000f} = {\cdot}00043\,(W + U)\,\frac{L}{f}.$$

This gives the weight of the girder to carry the dead and live loads. In addition to this there is the weight of the girder itself. The weight for this W_2 can be determined by simple proportion, thus:—

$$\frac{W_2}{W_1} = \frac{W_1}{W + U};$$

and the total weight of the girder is

$$W_{01} = W_2 + W_1 = W_1\left(\frac{W_1}{W + U} + 1\right).$$

This formula has been tested against the actual weights of girders, and, as will be seen from the enclosed table, gives good average results. For spans below 40 feet, the values obtained required to be slightly increased. This is to be expected, owing to the difficulty of designing small girders without a certain amount of waste of material.

WEIGHTS OF GIRDERS.

Total Length.	Total Depth.	Clear Span.	Live-load Tons.	Dead-load Tons.	Total Load.	Working Strength. Tons per sq. inch.	Weight, Tons.	Calculated Weight, Tons.
ft. in.	ft. in.	ft. in.						
38 0	5 0	32 6	72	35	107	4·5	5·3	5·0
38 0	5 0	33 0	75	37	112	4·5	5·0	5·1
38 0	5 0	33 0	75	44	119	4·5	6·0	5·6
39 6	5 0	35 0	75	41	116	4·5	6·0	5·6
41 0	5 0	36 0	78	41	119	4·5	6·0	6·0
45 6	5 0	40 6	84	50	134	4·5	7·5	7·5
95 6	10 6	89 0	164	106	270	4·5	31·0	33·0
100 0	11 6	94 0	162	122	284	4·5	38·0	37·0
106 0	12 0	100 0	170	92	262	4·5	42·0	36·5
111 0	11 6	103 0	176	127	303	4·5	46·0	44·0
111 6	11 6	105 0	177	138	315	4·5	46·0	47·0

It is interesting to note that, within wide limits, the values of r do not cause much variation in the volume of the girder. This is in accordance with practical experience. Generally, for moderate spans, it is easier to design the girder with values of r from $\frac{1}{8}$ to $\frac{1}{10}$, the web then being thicker, and giving a girder with better proportions than could be obtained with greater depths; and the girder as a whole is more easily stiffened. Considered theoretically, however, larger ratios could be taken with little or no greater volume of material in the girder.

It is shown in § 71 that, for economy, the volume required to transmit the horizontal stresses equals that required to transmit the vertical stresses, and that this relation is generally true, and for all kinds of loading. The depth of girders in practice is much less than the theoretical economic depth; and it is to be expected that the above relation would still hold true. Now, for such girders, the horizontal stresses transmitted by the web or bracing are relatively small, compared with the horizontal stresses transmitted by the flanges, and can be neglected; the above relation can then, with little error, be expressed by saying that for maximum economy in a girder the volume of the flanges should be equal to that of the web and the struts. The formula shows this when the value of r is about $\frac{1}{9}$ to $\frac{1}{8}$. It follows, if the economic depth is given by a certain value of r, that it is always possible to design two girders with a ratio of depth to span, one $> r$ and one $< r$, having equal volumes, each of which may be only slightly in excess of the economic volumes. This explains the reason why so much discussion has taken place with regard to the economic depths of girders, the fact being that there is a range about the economic depth, in which there is little variation from the economic volume.

It is to be noted that, whatever views may be taken with regard to the constants that have been adopted in the formula for the volume of the flanges, web, &c., any variation in these would not affect the general deductions alluded to, and would only affect the economic depths.

Generally speaking, it appears that in small spans the economic depth is small and increases as the span increases, varying from $r = \frac{1}{10}$ in small spans, to as much as $r = \frac{1}{6}$ in large spans for lattice and triangulated girders.

THE WEIGHT OF PARALLEL GIRDERS.

70. **Application of the Analysis to the Weights of Parallel Girders.**—The determination of the economic ratio of the depth to span for a parallel girder, together with the best system of triangulation to be adopted, is a problem which cannot be solved without taking into consideration the material required for stiffening the struts, and the booms and connections. It is, however, simpler to attack the problem by neglecting these latter quantities, and to consider the theoretical volume required for the booms and triangulation alone, and afterwards to take into consideration the material required for stiffening.

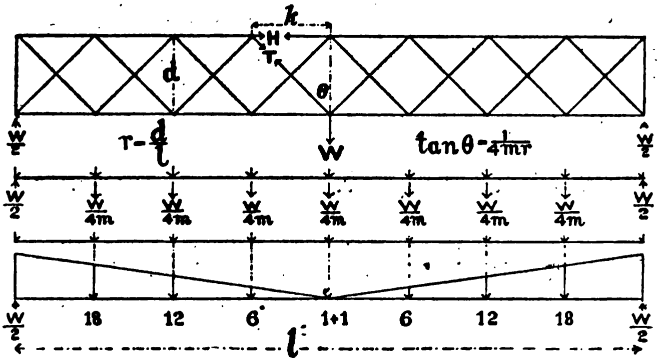


Fig. 63.

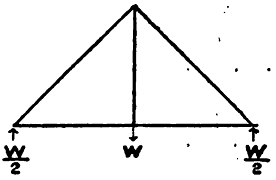


Fig. 64.

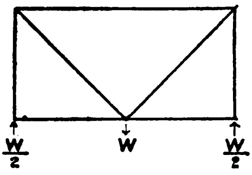


Fig. 65.

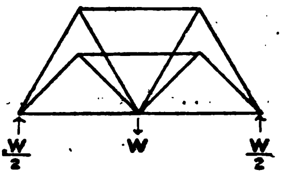


Fig. 66.

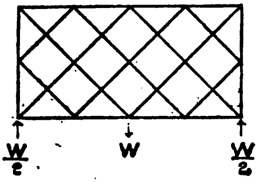


Fig. 67.

the girder; as in the case of the plate-girder, it is this latter quantity which exercises a dominating influence in determining the economic ratio of the depth to span of the girder. It will be at first assumed that the girders are rectangular, as shown on fig. 63—the analysis afterwards being extended to triangulated girders in which the end-struts are not necessarily vertical.

Concentrated Load at the Centre of Span.—A rectangular girder is shown on fig. 63 with a double system of triangulation carrying a concentrated load W at the centre of the span.

The expression for the volume V of the girder will be the sum of the volumes of the booms V_1, of the bracing V_2, and of the end-struts and vertical ties V_3.

Volume of Booms.—Assuming one-half the load to be carried by each system of triangulation and resolving the stresses along the bars; using the notation shown on figure, then

$$H = \frac{W}{4}\tan\theta = af; \quad \therefore a = \frac{W}{4f}\tan\theta = \frac{W}{16frm},$$

where $$\frac{d}{l} = r, \quad \tan\theta = \frac{l}{4md} = \frac{1}{4mr}, \quad k = \frac{l}{4m},$$

m being the even or odd number of bays in the half-span of the girder.

$$\text{Volume} \quad = v_1 = ka = \frac{W}{16frm}\cdot 2\cdot\frac{l}{4m},$$

$$v_2 = 2ka = \frac{W}{16frm}\cdot 2\cdot\frac{2l}{4m},$$

$$v_3 = 3ka = \frac{W}{16frm}\cdot 2\cdot\frac{3l}{4m},$$

$$\cdots\cdots\cdots\cdots$$

$$v_n = nka = \frac{W}{16frm}\cdot\frac{2ml}{4m};$$

$$\therefore \Sigma v = \tfrac{1}{4}V_1 = \frac{W}{16frm}\cdot\frac{l}{4m}\{2(1+2+3+\ldots+\overline{2m-1})+2m\}$$

$$= \frac{Wl}{16fr};$$

$$\therefore V_1 = \frac{Wl}{4fr}.$$

This shows that the volume of the booms is independent of the system of triangulation, and therefore independent of m.

Volume of Bracing.—Using the notation just given,

$$T = \frac{W}{4} \sec \theta = af; \quad \therefore\ a = \frac{W}{4f} \sec \theta.$$

$$\text{Volume of one bar} = al' = \frac{W}{4f} \sec \theta \, . \, \frac{l}{4m \sin \theta}$$

$$= \frac{Wl}{8fm \sin 2\theta},$$

and

$$\tfrac{1}{4} V_2 = 2m \, . \, \frac{Wl}{8fm \sin 2\theta},$$

$$V_2 = \frac{Wl}{f \sin 2\theta}.$$

The volume of the bracing is independent of the depth of the girder for a given value of θ, and is a minimum when $\sin 2\theta$ is a maximum, that is, when $\theta = 45°$.

Again,

$$\frac{1}{\sin 2\theta} = \tfrac{1}{2}(\tan \theta + \cot \theta) = \tfrac{1}{2}\left(\frac{1}{4mr} + 4mr\right).$$

This shows that the product of mr = a constant for a given system of triangulation.

Volume of End-Struts and Vertical Ties.—Each of the end-struts carries a load of $\frac{W}{4}$ and $a = \frac{W}{4f}$; and the volume is $ad = \frac{Wd}{4f}$, and for two $= \frac{Wd}{2f}$.

The vertical tie carries a load of $\frac{W}{2}$, giving a volume of $\frac{W}{2}\frac{d}{f}$; hence the volume of end-struts and vertical ties is

$$V_3 = \frac{Wd}{f} = \frac{Wlr}{f}.$$

Minimum Volume.—The total volume is $V = V_1 + V_2 + V_3$; collecting these values,

$$V = \frac{Wl}{f}\left(\frac{1}{4r} + \frac{1}{\sin 2\theta} + r\right).$$

This gives, on differentiation, as in the case of the plate-girder,

$r = \frac{1}{2}$ for the economic volume, treating the expression for the bracing as constant. If now the value $\frac{1}{2}\left(\frac{1}{4mr} + 4mr\right)$ be substituted for $\frac{1}{\sin 2\theta}$, and the value of m be put equal to $\frac{1}{2}$—that is, when one bay covers the whole span—then

$$V = \frac{Wl}{f}\left(\frac{1}{4r} + \frac{1}{4r} + r + r\right);$$

and, as before, on differentiation r is again $\frac{1}{2}$. Also $\frac{1}{\sin 2\theta}$ then equals 1, showing 45° for the minimum volume.

Figs. 64, 65, 67 are girders of equivalent volume for the concentrated load. The lattice-work shown on fig. 67 can theoretically be multiplied indefinitely without affecting the volume of the bracing. The result, however, is only of theoretical interest, as it neglects all consideration of the volume of material required for stiffening the girder.

By doubling over the inclined bars, as shown on fig. 66, a second girder is obtained with two bays in the span; and the analysis shows that the volume of this girder is the same as that just given; further, by putting $m = 1$ and differentiating, the girder of minimum volume is obtained and is made up of three equilateral triangles also shown on fig. 66. These girders are inadmissible from the point of view of this analysis, as they do not give the minimum volume when the triangulation is duplicated and inscribed in a rectangle as in fig. 67.

71. **General Statement for the Minimum Volume.**—From the analysis given in § 70 for the concentrated load, and from the results obtained for the plate-girder, the expression for the minimum theoretical volume—that is, neglecting stiffening—is always of the form

$$V = V_1 + V_2 + V_3,$$

where V_1 is the volume of the flanges, V_2 of the web or bracing, and V_3 of the end-struts and vertical ties. Now, V_1 is always of the form of $\frac{a}{r}$, V_2 of the form of

$$\frac{b}{\sin 2\theta} = \frac{b}{2}(\tan\theta + \cot\theta) = \frac{b}{2}\left(\frac{c}{r} + \frac{r}{c}\right),$$

and V_3 of the form of d/r, where a, b, c, and d are constants depending on the load and type of girder. Of the terms—

V_1 or a/r relates to those members transmitting horizontal stresses; that is, the booms or flanges.

V_2 or $b\left(\frac{c}{r}+\frac{r}{c}\right)$ relates to those members transmitting inclined stresses—that is, the inclined struts and ties.

V_3 or dr relates to those members transmitting vertical stresses—that is, end-struts and vertical ties.

It will be noticed that the term

$$V_2 \text{ or } b\left(\frac{c}{r}+\frac{r}{c}\right)$$

is of the form for members transmitting both horizontal and vertical stresses; that is to say, an inclined member may be looked upon as being made up of two members—one transmitting a horizontal stress, and one a vertical stress. This can be shown thus:—

Let an inclined member be given (fig. 68) transmitting a stress T; resolve the stress into two components H and W at

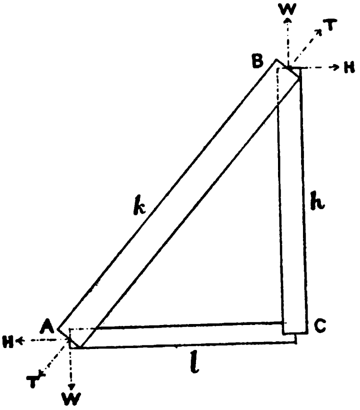


Fig. 68.

right angles to one another, and complete the right-angled triangle ABC. Then, from the figure,

$$T = W \sec \theta, \quad \text{and} \quad H = W \tan \theta;$$

also, $$l = k \sin \theta, \quad \text{and} \quad h = k \cos \theta.$$

The sectional area of the members k, l, and h will be

$$\frac{T}{f}, \quad \frac{H}{f}, \quad \text{and} \quad \frac{W}{f},$$

where f = the working-strength of the material. The volume of the members will be

$$\frac{Tk}{f}, \quad \frac{Hl}{f}, \quad \text{and} \quad \frac{Wl}{f}.$$

Substituting from the above equations, the volume of the member

$$k = \frac{Wl}{f} \cdot \frac{2}{\sin 2\theta};$$

and of the members l and h the volume is

$$\frac{Wl}{f}(\tan\theta + \cot\theta) = \frac{Wl}{f} \; \frac{2}{\sin 2\theta};$$

that is, the volume of the members l and h is equal to that of the member k. It follows, then, that any inclined member transmitting a stress may be looked upon as being made up of two members, one horizontal and one vertical, and whose joint volumes are equal to the volume of the inclined member. Now, it has been shown in the analysis, that

$$\tan\theta = \frac{1}{4mr}, \quad \text{and} \quad \cot\theta = 4mr;$$

that is, $\tan\theta + \cot\theta$ is of the form of $\frac{c}{r} + \frac{r}{c}$,

where c is a constant depending on the type of girder: hence,

$$\frac{Wl}{f}(\tan\theta + \cot\theta) = \frac{Wl}{f}\left(\frac{c}{r} + \frac{r}{c}\right);$$

and, putting $\frac{Wl}{f} = b$,

a constant depending on the distribution of loading and the length of the girder, the volume V_2 is of the form

$$b\left(\frac{c}{r} + \frac{r}{c}\right).$$

Collecting the coefficients for the total volume,

$$V = \frac{a}{r} + \frac{bc}{r} + \frac{b}{c}r + dr$$

$$= \frac{A}{r} + Br,$$

where $$A = a + bc, \quad \text{and} \quad B = \frac{b}{c} + a.$$

Differentiating and equating to zero for the minimum volume, the result obtained is

$$r = \sqrt{\frac{A}{B}}.$$

Putting this result in words:—*The economic volume is given when the volume of the members transmitting the horizontal stresses is equal to the volume of the members transmitting the vertical stresses.* This relation has not hitherto been remarked: it applies not only to girders, but to all types of engineering structures carrying vertical loads, and is perfectly general in its application.

72. **Uniform Load distributed over Span.**—The double triangulated girder is shown on fig. 63, with the uniform load equally distributed among the joints on the lower boom. The notation will be the same as that already given, and shown on the figure.

Volume of Booms.—Confining our attention to one system of triangulation, and starting from the bottom of the girder at the centre of the span:

$$\text{stress in top central bar} = H = \frac{W}{8m}\tan\theta = \frac{W}{8m}\cdot\frac{1}{4mr};$$

$$\text{cross-section of bar} = a = \frac{W}{8fm}\cdot\frac{1}{4mr};$$

$$\text{volume of bar} = ak = \frac{W}{32frm^2}\cdot\frac{l}{4m}\cdot 2.$$

Summing up the volumes for all the bars in a similar manner to that of the concentrated load, then the volume of one-half of the booms for the half load carried by this triangulation is

$$\frac{Wl}{128frm^3}[(1+1)1+(1+3)2+(3+3)3+(3+5)4 \text{ to } (2m-1) \text{ terms} + (2m-1)\,2m]$$

$$= \frac{Wl}{128frm^3}[2(1^2+2^2+3^2 \ldots . 2m-1 \text{ terms}) + 2m\,(2m-1)]$$

$$= \frac{Wl}{128frm^3}\left[\frac{(2m)(2m-1)(4m+2)}{3}\right]$$

$$= \frac{Wl}{24fr}\left(1-\frac{1}{4m^2}\right).$$

Summing up by a similar series, the volumes of one-half the booms for the half load carried by the second triangulation, the volume obtained is

$$\frac{Wl}{128frm^3}\left[\frac{2m\,(2m-1)(4m-1)}{3}+4m^2\right] = \frac{Wl}{24fr}\left[1-\frac{1}{4m^2}+\frac{3}{8m^2}\right].$$

Adding the two volumes together, and multiplying by two for both sides of the girder, the total volume is

$$V_1 = \frac{Wl}{6fr}\left[1-\frac{1}{16m^2}\right].$$

It is evident, from the formulæ just given, that for a girder made up of one system of triangulation alone, the volume of the booms is

$$V_1 = \frac{Wl}{6fr}\left(1-\frac{1}{4m^2}\right).$$

For a girder of the double triangulated or lattice type, or of the type when the load is partly suspended from the vertices of the triangle, as shown in fig. 69, No. 1, the volume of the booms is

$$V_1 = \frac{Wl}{6fr}\left(1-\frac{1}{16m^2}\right).$$

Volume of Bracing.—Confining our attention to one system of triangulation, and starting from the bottom of the girder at the centre of the span,

$$T = \frac{W}{8m} \sec \theta, \quad a = \frac{W}{8fm} \sec \theta.$$

$$\text{Volume of one bar} = al' = \frac{W}{8fm} \sec \theta \frac{l}{4m \sin \theta} = \frac{W}{16fm^2 \sin 2\theta}.$$

$$\tfrac{1}{2}\Sigma \text{ volumes} = \frac{Wl}{16fm^2 \sin 2\theta} . 2(1 + 3 + 5 \ldots \text{ to } m \text{ terms})$$

$$= \frac{Wl}{8f \sin 2\theta}.$$

For the second system of triangulation:—

$$\text{Volume of one bar} = \frac{Wl}{8fm^2 \sin 2\theta}.$$

$$\tfrac{1}{2}\Sigma \text{ volumes} = \frac{Wl}{8fm^2 \sin 2\theta} [2.(1 + 2 + 3 \ldots \text{ to } m - 1 \text{ terms}) + m]$$

$$= \frac{Wl}{8f \sin 2\theta}.$$

Adding these two volumes together, and multiplying by two for the total volume,

$$V_2 = \frac{Wl}{2f \sin 2\theta}.$$

Volume of End-Struts.—Each of the end-struts carries a load of $\frac{W}{4}$; their cross-section is $a = \frac{W}{4f}$, and the volume of one strut

$$= ad = \frac{Wd}{4f} = \frac{Wlr}{4f};$$

and for two, the total volume is

$$V_3 = \frac{Wlr}{2f}.$$

The vertical ties of the girder, shown on fig. 69, No. 1, carry one-half the total load; and the sum total of their volumes will be the same as for the end-struts just determined.

Minimum Volume.—The total volume is $V = V_1 + V_2 + V_3$. Collecting these values

$$V = \frac{Wl}{f}\left[\frac{1}{6r}\left(1 - \frac{1}{16m^2}\right) + \frac{1}{2\sin 2\theta} + \frac{r}{2}\right].$$

Putting $m = \frac{1}{2}$, this equation becomes identical with that for the concentrated load, giving, as before, $r = \frac{1}{2}$ for the minimum volume; and the girder takes the form given on figs. 64, 65.

Consider, now, that the girder remains of the lattice type shown in fig. 67; and that the bracing is indefinitely reduplicated, and that the load is transmitted to the bracing by vertical ties. The volume of these ties will be $\frac{Wlr}{2f}$; and the number of points at which the load is applied along the span can be anything up to infinity. In this case, the volume of the girder will be

$$V = \frac{Wl}{f}\left[\frac{1}{6r}\left(1 - \frac{1}{16m^2}\right) + \frac{1}{2\sin 2\theta} + r\right].$$

Putting in the value $m = \frac{1}{2}$, and

$$\frac{1}{2}\left(\frac{1}{4mr} + 4mr\right) \text{ for the } \sin 2\theta,$$

and differentiating, the minimum volume is given by the value of

$$r = \frac{1}{\sqrt{6}} = \frac{1}{2 \cdot 4},$$

or slightly less than the one-half previously obtained. As already remarked, these results are only of theoretical interest.

73. **Wedge Load distributed over Span.**—A pair of wedge loads are shown distributed over the span on fig. 63; it is required to determine the distribution of these loads at the joints along the bottom boom of the girder. Consider the load

to be divided up at the joints; then, by taking moments round the joints, the following formulæ are obtained:—

Let n = the number of divisions in the half-span = $2m$, W' = the weight of one wedge load, and R_0, $R_1 + R_2$, etc., the loads at the joints starting from the centre of the span; then

$$\begin{array}{lc|l} R_0 & = \dfrac{W'}{n^2} & \frac{1}{3} \\[2ex] \left.\begin{array}{l} R_1 \\ R_2 \end{array}\right\} & = & 2^2 - 1^2 - 1 \\[2ex] \left.\begin{array}{l} R_3 \\ R_4 \end{array}\right\} & = & 3^2 - 2^2 - 1 \\[2ex] \left.\begin{array}{l} R_5 \\ R_6 \end{array}\right\} & = & 4^2 - 3^2 - 1 \\[2ex] R_7 & = & 3 + \frac{2}{3}. \end{array}$$

The summation of the above gives the following series:—

$$W' = \frac{W'}{3n^2}\,[1 + 3 \,.\, 2 \,.\, (1 + 2 + 3 \ldots n - 1) + 3(n-1) + 2]$$

$$= \frac{W'}{3n^2}\left[1 + \frac{6(n-1)(n)}{2} + 3(n-1) + 2\right].$$

The shear stress at any joint is

$$F = \frac{W'}{3n^2}\Big[1 + 3(u-1)\,u\Big]_{u=n-1}^{u=0}$$

Load at centre joint $= \dfrac{W'}{3n^2}$.

Loads at joints $u = 1$ to $u = n - 1 = \dfrac{2W'u}{n^2}$.

The distribution of the loads shown at the joints on the figure can readily be verified from the above formulæ.

Volume of Booms.—The loads at the joints having been determined, the volume for these can be determined in a similar manner to that adopted for the uniform load.

Put $W = 2W'$, and $2m = n$; then the volume of the top central bar is

$$\frac{W}{2 \,.\, 3 \,.\, 4fm^2} \cdot \frac{1}{4mr} \cdot \frac{l}{4m}.$$

Summing the volume of the bars for one-half, the value obtained is

$$\tfrac{1}{2}V_1 = \frac{Wl}{384frm^4}\left[1+(6+1)(1+2)+(6+13)(2+3)+(13+24)(3+4)\right] \\ + \text{etc.}$$

$$= \quad ,, \quad \left[1+(2^3-1^3)3+(3^3-2^3)5+(4^3-3^3)7\right] \\ + \text{etc. to } n-1 \text{ terms}$$

$$= \quad ,, \quad \left[+(2n-1)n^3-2\left\{\frac{(n-1)(n)}{2}\right\}^2\right]$$

$$= \quad ,, \quad \left[\frac{n^2}{2}(3n^2-1)\right] = \frac{Wl}{384frm^4}\left[\frac{4m^2}{2}.12m^2\left(1-\frac{1}{12m^2}\right)\right]$$

$$V_1 = \frac{Wl}{8fr}\left(1-\frac{1}{12m^2}\right).$$

Volume of Bracing.—Consider both systems of triangulation together, starting from the lower boom of the girder at the centre of the span. The tension on the first bar is

$$T = \frac{W'}{3n^2}\sec\theta = \frac{W}{24m^2}\sec\theta,$$

where $W' = \frac{W}{2}$, and $n = 2m$, and $a = \frac{W}{24fm^2}\sec\theta$.

As previously, $l' = \frac{l}{4m\sin\theta}$;

hence, the volume of the first bar is

$$v = al' = \frac{W\sec\theta}{24fm^2}\cdot\frac{l}{4m\sin\theta} = \frac{Wl}{48fm^3\sin 2\theta}.$$

Summing up, the volumes for

$$\begin{array}{ll} \text{the first bar} & = \dfrac{Wl}{48fm^3\sin 2\theta} \\ ,, \ \text{next two} & \\ ,, \quad ,, & \\ ,, \quad ,, & \end{array} \left[\begin{array}{r} 1 \\ 6+\ 1 \\ 12+\ 7 \\ 18+19 \end{array}\right] = \left[\begin{array}{l} 1^3 \\ 2^3-1^3 \\ 3^3-2^3 \\ 4^3-3^3 \end{array}\right]$$

and so on to $2m$ terms. The sum of this series is $8m^3$, giving for the half-volume of the bracing

$$\tfrac{1}{2} V_2 = \frac{Wl}{48fm^3 \sin 2\theta} \cdot 8m^3,$$

and

$$V_2 = \frac{Wl}{3f \sin 2\theta}.$$

Volume of End-struts and Ties.—For the distribution of the loads at the joints, shown on fig. 63, the load on the end-strut is

$$\frac{W'}{3n^2} \cdot 6 \left(\frac{n}{2}\right)^2 = \frac{W'}{2} = \frac{W}{4}:$$

hence,

$$a = \frac{W}{4f}, \quad \text{and the volume} \quad = \frac{Wd}{4f} = \frac{Wlr}{4f},$$

and for two struts

$$= \frac{Wlr}{2f}.$$

As in the case of the uniform load, assume the loads to be transmitted to the bracing by vertical ties, their volume will be, as before,

$$\frac{Wlr}{2f}, \quad \text{giving} \quad V_3 = \frac{Wlr}{f}.$$

Minimum Volume.—

$$V = V_1 + V_2 + V_3$$

$$= \frac{Wl}{f} \left[\frac{1}{8r} \left(1 - \frac{1}{12m^2}\right) + \frac{1}{3 \sin 2\theta} + r \right].$$

Putting $m = \frac{1}{2}$, and substituting for $\sin 2\theta$, as before, the value obtained after differentiation for r is $\frac{1}{\sqrt{8}}$, which, as is to be expected, is less than the value obtained for the uniform load.

74. **Live and Dead Loads.**—From the inspection of the formulæ for the volumes of the bracing, it will be noted that, under concentrated, uniform, and wedge loads, respectively, the volumes are in the proportion of $1 : \frac{1}{2} : \frac{1}{3}$, that is, proportional to the area of the shearing-force diagrams due to the different loads. This relation was also shown to be true for the

plate-girder: hence, it follows generally for plate, lattice, or triangulated girders.

In § 68 a table was given, showing the volume of the web for the plate-girder under live and dead loads; and from the above statement, the table applies equally to the volume of the ties and struts of lattice and triangulated girders. Using the values there given,

$$V_2 = \frac{l}{f}\left(\frac{W}{2} + \frac{7}{12}U\right)\frac{1}{\sin 2\theta},$$

$$V_3 = \frac{l}{2f}(W + U)r.$$

The volume of the booms will be

$$V_1 = (W + U)\frac{l}{f}\left\{\frac{1}{6r}\left(1 - \frac{1}{gm^2}\right)\right\},$$

the value of g being 4 or 16 according as the simple triangulated or lattice-girder is under consideration. For all practical cases the value of $\frac{1}{gm^2}$ is so small, usually under 1 per cent., that it can be neglected without sensible error. The formula given for the volume of the booms assumes the live and dead loads uniformly distributed, as under these conditions the stresses in the booms are a maximum.

Summing up the values for the total volume

$$V = \frac{l}{f}\left[(W + U)\frac{1}{6r} + \left(\frac{W}{2} + \frac{7}{12}U\right)\frac{1}{\sin 2\theta} + (W + U)\frac{r}{2}\right],$$

this formula is nearly the same as

$$V = (W + U)\frac{l}{f}\left[\frac{1}{6r} + \frac{1}{2}\frac{1}{\sin 2\theta} + \frac{r}{2}\right],$$

that is, the volume for a girder on which the loads are uniformly distributed; and it is a sufficiently near approximation for all cases that arise in practice.

Let the moving load be a multiple of the dead load, that is, put $U = kW$. Substituting in the above equation

$$V_0 = \frac{Wl}{f}\left[(1 + k)\left(\frac{1}{6r} + \frac{r}{2}\right) + \left(\tfrac{1}{2} + \tfrac{7}{12}k\right)\frac{1}{\sin 2\theta}\right].$$

Application of the Analysis to Parallel Girders.—A number of girders of different types are shown on fig. 69, with parallel booms and various types of bracing, the ratio of the depth to span being $\frac{1}{8}$ throughout. The volume of the booms will be sensibly

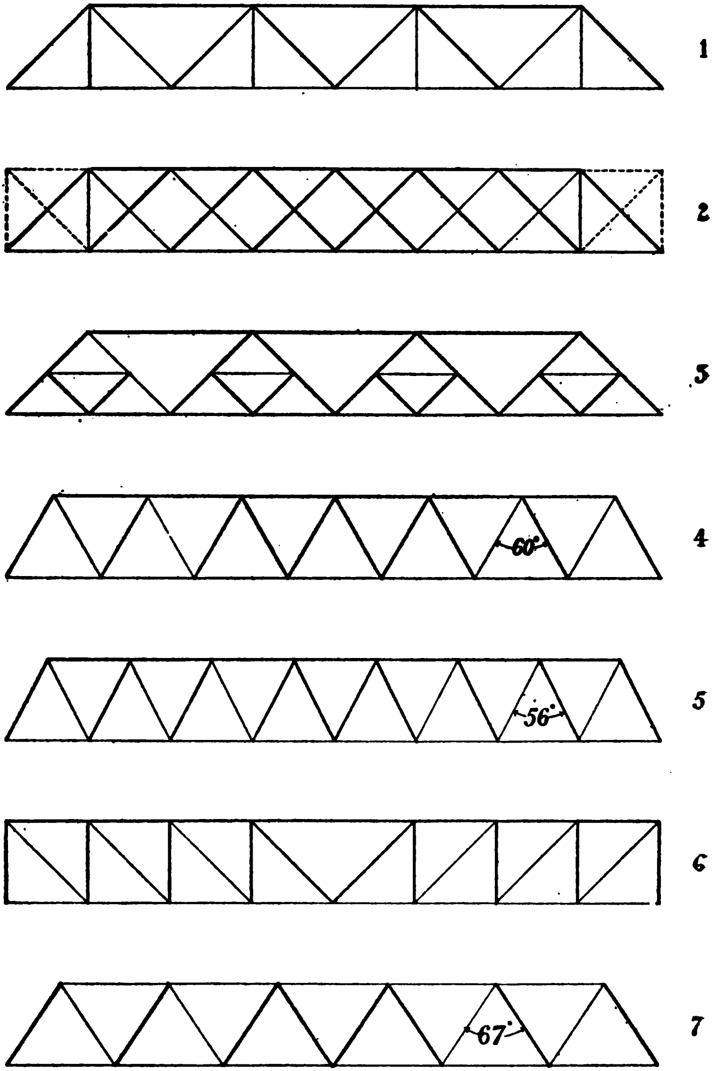


Fig. 69.

the same for all and equal to $\frac{(W+U)l}{6fr}$. The volumes of the bracing and end-struts are given in the accompanying table. In making the comparison between different systems of bracing, *that system which for a given volume gives the greatest number of bays in the span is the most efficient.* The table shows conclusively the economy of the 45° bracing when judged from this point of view.

The value of k has been assumed to be 2 throughout in the above formula, being a fair average value of the ratio of the moving to the dead load for spans of 100–120 feet. The formula then takes the form

$$V_0 = \frac{3Wl}{f}\left\{\frac{1}{6r} + \frac{5}{9}\,\frac{1}{\sin 2\theta} + \frac{r}{2}\right\}.$$

$$\text{Volumes} = \frac{3Wl}{f} \times \text{coefficient in table.}$$

Figure.	Bracing.	End-struts.	Total.	Bays in span.
1	·55	·06	·61	8
2	·55	·06	·61	8
3	·55	·06*	·61	8
4	·64	—	·64	7
5	·66	—	·66	8
6	·83	—	·83	8
7	·60	—	·60	6

With the value of $r = \frac{1}{8}$ there is no difficulty in arranging for the bracing at 45°, and this is also true for values of $r = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$, &c. When, however, r has some intermediate

---

* The volume of the booms in the supplementary girders. For the right-angled truss of No. 6, the volume of the bracing

$$= \frac{5}{9}\left(\frac{1}{\sin 2\theta} + \frac{\cot\theta}{2}\right);$$

the end-struts are included in this expression, as in the case of the plate-girder.

value, it is better to arrange the bracing at some slightly different angle, keeping it as near 45° as possible, if the maximum economy of material is to be obtained.

The comparison of the volumes of the bracing shows that a considerable variation from the economic angle 45° does not lead to a corresponding increase in the volume of the bracing; for instance, on comparing the girders Nos. 1 and 5 (fig. 69), the increase is about 8 per cent. As, however, the sine and cotangent rapidly change in value with the angle below 28°, the increase in the volume becomes very marked, amounting to as much as 33 per cent. for the type of girder shown on No. 6 (fig. 69).

75. **Volume required for Stiffening.**—From the examination of the formulæ obtained for the theoretical volumes of girders under different loads, and from § 71, it has been proved that the minimum volume is given when the volume of the members transmitting the horizontal stresses is equal to the volume of the members transmitting the vertical stresses. For the economical design of parallel girders, this relation will always hold true. Now, for the usual values of r, the volume of the members transmitting the horizontal stresses is small; and the relation may be more tersely expressed, that for economy the volume of the booms equals that of the bracing.

Since in practice it has been found that the economic depths are much less than the theoretical economic depths, the question arises of the influence of the coefficients for stiffening, and what assumptions must be made for their determination. It has been pointed out, when considering the plate-girder, that for the struts the coefficient varies as r^2 if the value of ρ, the radius of gyration, is assumed to remain constant. With little error this will also apply to braced girders, since, for any particular span, the value of ρ practically remains constant for any small change in the depth.

Owing to the load not acting truly in the vertical plane of the girder, there is a moment tending to distort the girder, and which will vary approximately as the length of the girder. If the area of the stiffener to resist this moment varies as the depth, the distortion of the girder will vary as the square of the depth; and the volume of the stiffener will be proportional to r^2.

For the wind-pressure the coefficient will vary as r^2; it has, however, nothing to do with the load. Approximately it may be assumed to depend on $W + U$; and, if so, it can then be included in the general expression for the volume.

Treating the load on the girder as a uniform load the expression for the volume will be of the form

$$V_0 = \frac{Wl}{f}\left(\frac{a}{r} + br + cr^2\right),$$

where a, b, and c are constants. Of these, a and b admit of being closely determined for any particular type of girder. The value of c must be determined empirically, and depends upon the assumptions made for stiffening, and will depend on the type and span of the girder.

As a first approximation, the formulæ given for the volume of plate-girder in § 69 may be used in the following form:—

$$W_1 = \frac{(W + U)L}{8000f}\left(\frac{1}{5r} + 1 + Ar^2\right);$$

the coefficient A having the following values:—

for 100′ spans, ...	50,
„ 200′ „	45,
„ 300′ „	40,

interpolated values being used as required.

The total weight of the girder will be

$$W_0 = W_1\left(\frac{W_1}{W + U} + 1\right),$$

the symbols having the same significations as already given for the plate-girders.

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THE DESIGN
OF
COLUMNS AND STRUTS.

ERRATA.

THE DESIGN OF COLUMNS.

Page 29—For the square section $N = 40$ *read* 60.
„ „ cross „ $N = 20$ *read* 120.
„ „ triangular „ $N = 30$ *read* 80.
„ „ I „ $N = 35$ *read* 70.

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PREFACE.

THE following papers on "The Design of Columns and Struts," "The Economic Design of Columns," and on "Web Stresses in Plate Girders" in the present form are the result of four years' research work in this direction, and the thanks of the author are due to the Institution of Civil Engineers of Ireland and to *Engineering* for permission to republish them.

The paper on "Web Stresses in Plate Girders" bears directly on the matter given in the author's work on "The Design of Plate Girders," and the views therein expressed are in general borne out by the experimental research work. Section 58 requires some modification, but no further important corrections appear to be necessary.

W. E. LILLY.

THE DESIGN OF COLUMNS AND STRUTS.

THE problem usually involved in the design of a column or strut is:—Given a definite load and length, to determine its cross-section. The solution of the problem is complex in that it involves the length and the figure or shape of the cross-section and the proportions thereof, together with the way in which the load is applied to the column. It is now proposed to consider, when viewed in the light of recent research work in this direction, this problem, together with some of the principles which underlie the strength of columns and struts, and to put forward some formulas by means of which the usual problems that occur in their design may be solved.

SOLID CIRCULAR ROUND-ENDED COLUMNS.

In the accompanying Fig. 1 are shown a series of curves which have been obtained from tests carried out on a large number of round bars of ½-inch diameter of different materials. The mode of testing was in all cases the same—definite lengths of the bars were cut off and the ends rounded in the lathe, the bars being at the same time carefully straightened. They were then placed vertically in the testing machine, the round ends bearing on steel discs, the surfaces of which were slightly concave to enable the specimen to be easily adjusted. The test was then carried out in the usual way, and the load producing failure noted. The manner of recording the

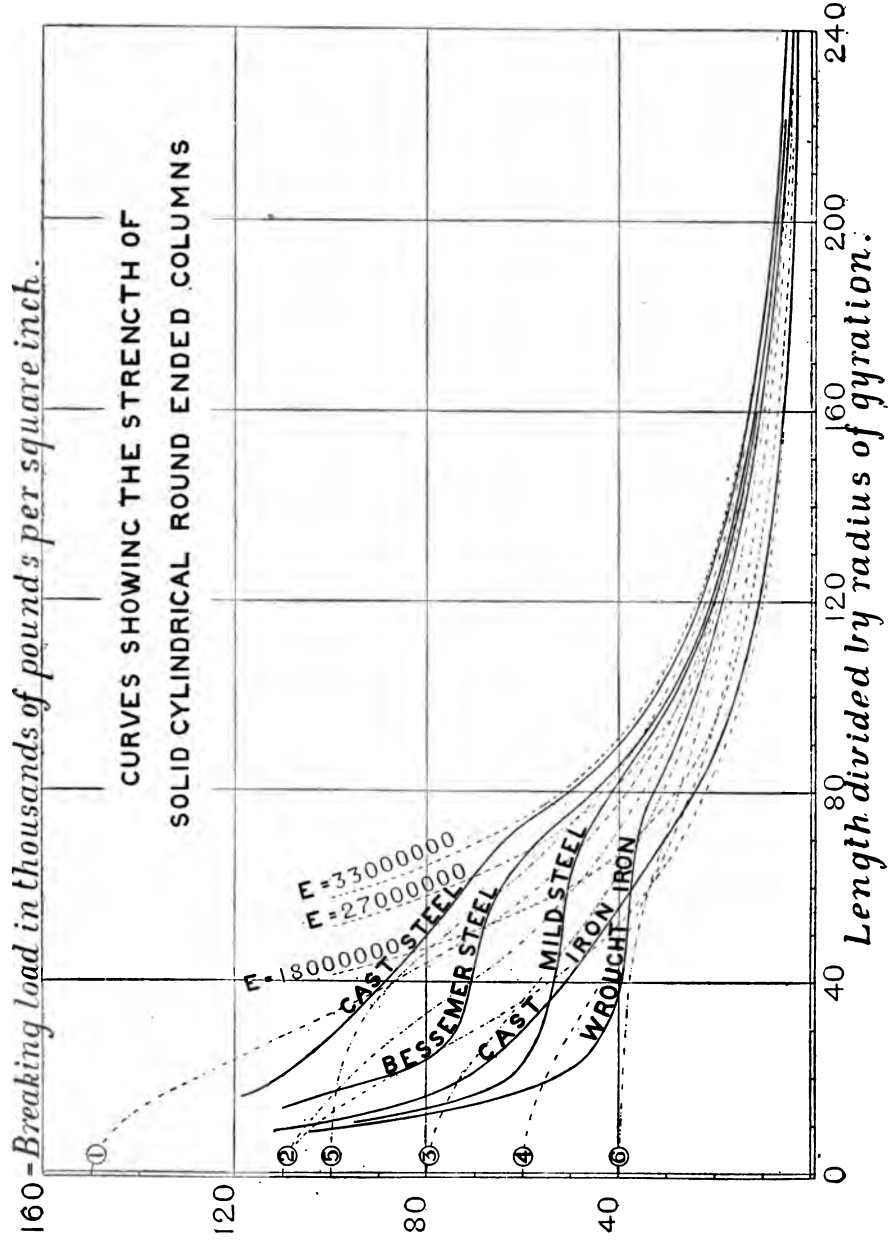
CURVES SHOWINC THE STRENCTH OF
SOLID CYLINDRICAL ROUND ENDED COLUMNS
Breaking load in thousands of pounds per square inch.
Length divided by radius of gyration.
160
120
80
40
0
40
80
120
160
200
240
E=33000000
E=27000000
E=18000000
CAST STEEL
BESSEMER STEEL
MILD STEEL
CAST IRON
WROUCHT IRON
①
②
③
④
⑤
⑥

Fig. 1.

results on the diagram was as follows:—Values of l/ρ, the length divided by the radius of gyration, were plotted on a horizontal line as base, and the values of p, the load in pounds per square inch on the column, were plotted vertically; the curves were then drawn through the points so obtained. The author would have had some diffidence in recording the results of these experiments in that they were carried out on round bars of the small diameter of $\frac{1}{2}$-inch; the values obtained were, however, carefully scrutinised and compared with the values obtained by other experimentalists, and found to be in close agreement, so much so that they afford direct evidence that the influence of size is not of great importance when comparing the strength of columns. The test made by the Commission on the model chord member of the Quebec Bridge may be cited in support of this view, and also various other published tests recorded by J. M. Moncrieff in the "Transactions of the American Society of Civil Engineers," 1900, Vol. XLV.

The tests were carried out on samples of cast tool steel, Bessemer steel, mild steel shafting annealed and unannealed, wrought iron and cast iron. The curves obtained are shown on Fig. 1, and the stress strain diagrams from the tension tests, together with the values for Young's modulus of elasticity, being given on Fig. 2. It will be noted that for values of l/ρ greater than 120 there is little difference between the values of the breaking loads obtained on the various materials. For values beyond this limit the modulus of elasticity governs the strength of all the columns, and the values closely approximate to the dotted curves plotted to Euler's formula. For values of l/ρ less than 120 and greater than 40, the ductility and the strength of the material at the

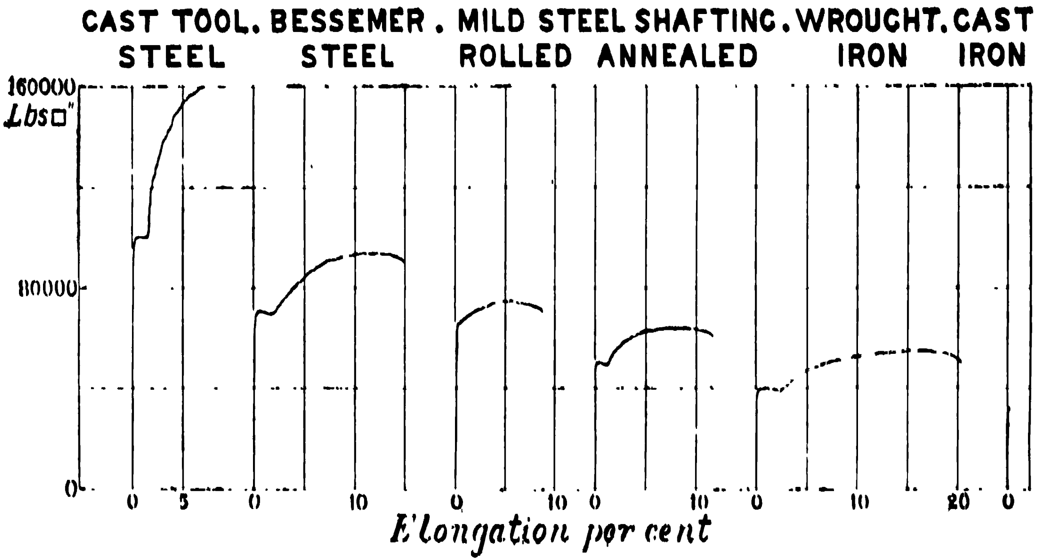

Fig. 9.

Breaking Load— Pounds per square inch	159,000	93,500	74,000	64,000	55,000	33,000
Load at Yield Point	100,000	70,000	64,000	50,000	39,000	—
Young's Modulus E	33,000,000	31,000,000	30,000,000	30,000,000	28,000,000	18,000,000

elastic limit or near the yield point play an important part in determining the strength of the columns; the load producing failure partly stresses the material beyond the elastic limit, and, on taking off the load, the column is found to have received a permanent set or deformation.

In the case of the cast tool steel the load at the yield point was nearly 100,000 pounds per square inch, and the elongation 5.6 per cent. With such a material it is not until values of l/ρ less than 70 are reached that any marked defect from the Eulerian curve is observed. The breaking load then stresses the material beyond the elastic limit and causes a variation of the bending strains on the cross-section, which can no longer be said to follow the laws of flexure; also, owing to the ductility of the material being small, the strength of the column for values of l/ρ less than 70 rises continuously till about the value of l/ρ equals 20 is reached, when direct crushing or flow of the material commences, and the load producing failure rises to enormous values.

Consider now the curve obtained for the wrought iron columns. For values of l/ρ less than 120 the defect from Euler's curve begins to be apparent, and the curve falls away and becomes much flatter than in the case of the cast steel columns, also the values obtained for the breaking loads are much less. The load at the yield point was 39,000 pounds per square inch, as shown on the stress strain diagram, and the elongation 21 per cent. Permanent deformation of the columns was observed for values of l/ρ about 120, the fibre stresses on the central cross-section then pass the elastic limit, and the curve of breaking loads now falls farther and farther away from the Eulerian curve till the value of l/ρ equals 80 is reached, when the ductility of the material greatly influences the strength of the columns. For values of l/ρ less than 80 and greater than 30 no great variation in their strength is apparent, and for values less than 30 direct crushing or flow of the material takes place and the loads causing failure rise to very great values.

From an examination of the curves obtained for Bessemer steel and mild steel it will be observed that similar remarks apply, although in a less marked degree than for the wrought iron, and when examined conjointly with the stress strain diagrams it appears that it is the strength of the material at the yield point and the percentage elongation which play a predominating part in determining the strength of the column between the values of l/ρ less than 120 and greater than 30. The curves obtained for the annealed and unannealed mild steel columns are of interest when considered from this point of view. The sample obtained was that of a rolled bright mild steel shafting, and the first series of tests were carried out on the sample as received. The columns were then carefully straightened, annealed, and again tested, giving a curve nearly coincident with that obtained for the wrought iron. It will be noted, when considered jointly with the stress strain diagrams, that the difference in the curves for the breaking load is sufficiently accounted for when the strengths of the material at the yield point in the annealed and unannealed states and the respective elongations are considered. It has previously been observed that if a column which has been tested and bent be straightened and given a good hammering, so as to raise the yield point and decrease the ductility, that the column on being re-tested will be stronger than it was in its previous condition. This is in accordance with the results of these experiments, but it only produces this marked result for values of l/ρ less than 120 and greater than 30—that is, on those parts of the curve where the ductility of the material and the strength at the yield point play a predominating part in determining the strength of the column. It was, doubtless, a knowledge of these facts that led the late Sir Benjamin Baker to specify a steel with a high elastic limit for the compression members of the Forth Bridge.

The curve obtained from the tests on the cast iron columns is of interest, being carried out on a set of bars obtained from one pouring of the metal and afterwards turned in the lathe to $\frac{1}{2}$-inch diameter. The author has not been able to find any recorded tests carried out on cast iron in which the specimens were turned before testing. The results obtained have been compared with Hodgkinson's experiments on round cast iron columns and found to be in close agreement, thus showing that the skin of the casting has no great effect on the strength of the column.

The sample of cast iron used was above the average, giving for the tensile test 33,000 pounds per square inch, and a crushing strength of 110,000 pounds per square inch, and for Young's modulus 18,000,000. The curve obtained displays a similarity in its shape to that obtained for the cast steel. When the small amount of the ductility of cast iron is considered the result is not surprising.

From a comparison of the curve given for cast iron with the other curves it will be noticed that at a value of $l/\rho=60$ the curves of wrought iron and cast iron intersect, and, therefore, for values less than, equal to, or greater than this value a solid column of cast iron is of a strength greater than, equal to, or less than one of wrought iron. Similar remarks also apply for the intersection of the curves for mild steel and cast iron at the value of $l/\rho=35$.

The other curves for the Bessemer steel and cast tool steel do not intersect the cast iron curve, and, therefore, columns of these materials are always stronger than those of cast iron.

The above relation for wrought iron and cast iron was first pointed out by Gordon. His estimate for the critical value of l/ρ was somewhat greater than that given above, being about 80. This is due to the columns having various shapes of the cross-section, the average curve of which intersects at a value of l/ρ greater than that obtained on solid circular columns.

In the testing of long columns some elastic deflection was always apparent before the load producing failure was reached. It has hitherto usually been assumed that once the critical load on the column was reached, as given by Euler's formula, the deflection was independent of any variation of the load, and, further, no deflection would be observed until the load reached the critical value. To account for the experimental fact that under much smaller loads a deflection of the column had been observed, it has been assumed that the deflection was due to the column not being straight or to want of homogeneity of the material. The author cannot agree with this view. The load which produces failure certainly agrees closely with the calculated values obtained by Euler's formula, but the deflection under such loading is considerable, and, therefore, it is better to assume that the load producing failure will bend the column to its proof deflection. For loads less than the limiting load some deflection is always apparent, and it is curious to note the range of what may be called the resilient deflection in testing such columns. With loads less than the critical load the column may be pushed in the middle, and on releasing the push it will recover itself, but will not become straight. The deflection observable increases with the load, or, in other words, the deflection is some function of the load, other things being constant, and the author believes this to be true for the ideal column. The result enunciated by Euler's analysis has in this respect had a retarding influence on the investigations on columns.

Usually it is considered that a solid column of a square or other figure of cross-section would give as good results as a column of a circular cross-section. The author finds that this is not the case. For values of l/ρ greater than 120 very little difference can be observed, but for values of l/ρ less than 120 and greater than 30 the shape or figure of the cross-section influences the strength of the column, and the values obtained are somewhat less than those obtained for the circular column. The experiments show that there is some analogy between the results obtained by St. Venant on the strength to torsion of shafts of various cross-sections compared with the strength to compression of such sections. The phenomena of secondary flexure have a direct bearing on this result.

Various formulas have been proposed at different times, based both upon experimental and theoretical grounds, to determine the strength of columns and struts. Of these the author considers the Rankine-Gordon formula to be the best. This formula is of a form proposed by Tredgold on theoretical grounds. It was afterwards revived by Gordon, who determined the approximate values of the constants, and then modified by Rankine, who substituted the value of ρ the radius of gyration for the diameter of the cross-section. The formula only applies to solid columns, and requires modification when applied to hollow or irregular forms of cross-section.

The Rankine-Gordon Formula.

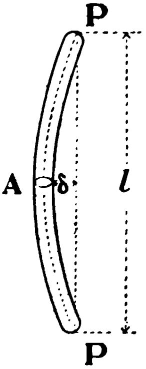

Fig. 3.

The formula may be derived as follows:— Consider the column of uniform cross section shown on Fig. 3, and assume that the column is slightly bent by the load P, then the bending moment at the middle is—

$$M=P\delta=pA\delta \qquad (1)$$

where P=the load on the column,
δ=the central deflection,
p=the load per unit of area,
A=the area of the cross section.

From the theory of flexure

$$\frac{M}{I}=\frac{p_{,}}{y} \qquad (2)$$

where I=the moment of inertia of the cross section about a diameter,
y=the distance of the outer fibre from the neutral axis,
$p_{,}$=the intensity of the outer fibre stress due to bending.

From equations (1) and (2) the following relation is obtained:—

$$pA\delta=\frac{p_{,}I}{y} \text{ or } \frac{p_{,}}{p}=\frac{\delta y}{\rho^2} \qquad (3)$$

Since $I=A\rho^2$ where ρ is the radius of gyration of the cross-section.

A further deduction from the theory of flexure shows that the proof deflection of a beam varies as $\frac{l^2}{y}$ or $\delta=c\frac{l^2}{y}$ where c is some constant. Substituting this value of δ in equation (3) the relation obtained is

$$\frac{p_{,}}{p}=c\frac{l^2}{\rho^2} \qquad (4)$$

Add unity to both sides of this equation, then

$$\frac{p_{,}+p}{p}=1+c\frac{l^2}{\rho^2}$$

Now $p_{,}+p$ equals the sum of the stresses due to the load and to bending and is equal to f the working strength to compression of the material, also $pA=P$. Substituting and reducing the Rankine-Gordon formula is obtained

$$P=\frac{fA}{1+c\frac{l^2}{\rho^2}} \qquad (5)$$

The constants as given by Rankine for wrought iron columns are

$f=36,000$ pounds per square inch,
$c=\frac{1}{9000}$ for columns with round ends,
$=\frac{1}{36000}$ for columns with fixed ends.

The manner of deriving the formula is open to some criticism in these respects. It assumes the deflection of a column to be the same as that of a beam, whereas the deflection of a column when it comes to its proof stress is less than that of a beam compared under similar conditions; also, the value of c is supposed to remain constant, which is not true if the sum $p + p_{,}$ is constant. This may be shown as follows:—

The equation for the central deflection of a beam is:—

$$\delta=\frac{Wl^3}{nEI} \qquad (6)$$

where W=the load on the beam,
l=the length of the beam,
E=Young's modulus of elasticity,
n=a constant.

For beams similarly and symmetrically loaded the equation takes the form

$$\delta=\frac{Ml^2}{mEI}$$

Where M is the maximum bending moment at the centre of the beam, and m a constant. For the case of a beam which is loaded so that the curve of deflection is sinusoidal, the formula for the deflection becomes

$$\delta = \frac{\omega \delta l^4}{\pi^4 E I} \qquad (7)$$

where ω is the intensity of the load per unit of length on the beam, and $M = \frac{\omega \delta l^2}{\pi^2}$.

Now let a column similar to the beam be so loaded as to have the same deflection as the beam, then

$$P\delta = M = \frac{\omega \delta l^2}{\pi^2},$$ and on substitution in equation 7

$$\delta = \frac{P\delta l^2}{\pi^2 E I} \text{ or } P = \frac{\pi^2 E I}{l^2}$$

which is Euler's formula. As previously

$$\frac{M}{I} = \frac{p_{,}}{y} \text{ or } \frac{P\delta}{I} = \frac{p_{,}}{y}$$

Now, if the cross-sections compared are kept similar and under similar loading $\frac{p_{,}}{y}$ can be substituted for $\frac{P\delta}{I}$ and the equation for the proof deflection becomes

$$\delta = \frac{p_{,} l^2}{\pi^2 E y} \qquad (8)$$

In a beam $p_{,}$ is the proof strength of the material, whereas in a column it is the sum of $p + p_{,}$ which should be the proof strength. If now the value of δ in equation 8 is substituted for δ in equation 3, then

$$\frac{p_{,}}{p} = \frac{p_{,} l^2}{\pi^2 E \rho^2} \text{ or } p = \frac{f}{1 + \frac{p_{,} l^2}{\pi^2 E \rho^2}} \qquad (9)$$

and the equation returns again to Euler's formula, but if, on the other hand, $p_{,}$ is put equal to f on the right hand side the form of the Rankine-Gordon formula becomes

$$p = \frac{P}{A} = \frac{f}{1 + \frac{f}{\pi^2 E} \frac{l^2}{\rho^2}} \qquad (10)$$

So far as long columns are concerned the value of f may be taken at anything at pleasure within large limits, as the formula always returns to the Eulerian values for long columns. For columns of medium and short lengths the value of f is so chosen as to fit in with the curves obtained from experiments. The best values to be adopted for different materials will be referred to presently. It will be admitted from the foregoing that the form of the Rankine-Gordon formula is more or less irrational in that the deflection of the column is assumed to be too large, more especially in columns of medium and short length, and that the sum of the stresses $p+p_{,}$ are not constant. The following variation of the formula, in which allowance is made for these defects, the author believes to be new. Reverting back again to equation 9, the value of $p_{,}$ can be obtained in a similar manner to that of p, giving

$$p_{,}=\frac{f}{1+\dfrac{\pi^2E\rho^2}{p_{,}l^2}}$$

Now substitute the value for $p_{,}$ in equation 9; the formula then becomes

$$p=\frac{f}{1+\dfrac{fl^2}{\pi^2E\rho^2}\left(\dfrac{1}{1+\dfrac{\pi^2E\rho^2}{p_{,}l^2}}\right)} \qquad (11)$$

and in a similar way it can be shown that

$$p_{,}=\frac{f}{1+\dfrac{\pi^2E\rho^2}{fl^2}+\left(\dfrac{\pi^2E\rho^2}{l^2}\right)^2\dfrac{1}{fp_{,}}} \qquad (12)$$

If now in equation 11 $f-p$ be substituted for $p_{,}$, and p be put equal to its Eulerian value, the equation returns to Euler's formula, but if $p_{,}$ be put equal to f so that p is equal to 0 the formula becomes

$$p=\frac{P}{A}=\frac{f}{1+\dfrac{fl^2}{\pi^2E\rho^2}\left(\dfrac{1}{1+\dfrac{\pi^2E\rho^2}{fl^2}}\right)} \qquad (13)$$

The formula so obtained with suitable values of f more closely corresponds to the experimental curves. Furthermore, it will be shown it is possible to fix three points on the curve with a considerable degree of accuracy by experiments on columns, and, therefore, the curve so obtained cannot be far from the true curve.

Reverting to equation 3, and substituting the values obtained for p and $p_{,}$ in equations 11 and 12, the relation obtained is

$$\frac{\delta y}{\rho^2} = \frac{p_{,}}{p} = \frac{\left(\dfrac{fl^2}{\pi^2 E\rho^2}\right)^2}{1+\dfrac{fl^2}{\pi^2 E\rho^2}} \qquad (14)$$

Let $\delta_{,} = \dfrac{fl^2}{\pi^2 Ey}$ = the deflection of a beam bent to a sine curve and δ = the deflection of a column similar to the beam bent to a sine curve then the above relation can be expressed

$$\frac{\delta_{,}}{\delta} = 1 + \frac{\pi^2 E\rho^2}{fl^2}$$

Some further remarks on the derived form of this formula will be found in the Appendix.

The constants to be used in the Rankine-Gordon formula, given in equation 10, now require to be determined; on the diagram four curves, numbered 1, 2, 3, 4, are shown plotted to the constants given in the following table:—

Material	Strength to compression f	Modulus of elasticity E	Round ended columns c	Fixed ended columns c
Cast Tool Steel	150000	33000000	$\frac{1}{2200}$	$\frac{1}{8800}$
Bessemer Steel	110000	31000000	$\frac{1}{2800}$	$\frac{1}{11200}$
Mild Steel -	80000	30000000	$\frac{1}{4000}$	$\frac{1}{16000}$
Do. Annealed -	60000	30000000	$\frac{1}{3000}$	$\frac{1}{20000}$
Wrought Iron -	55000	26000000	$\frac{1}{4500}$	$\frac{1}{18000}$
Cast Iron -	110000	14000000	$\frac{1}{1300}$	$\frac{1}{3200}$

The curve for wrought iron closely approximates to that for the annealed mild steel, and has not been drawn. It is to be understood that the values of f adopted have been so chosen as to give the nearest approximation to the experimental curves.

This of necessity must be the case when the manner of deriving the formula is considered. The object to be attained is to make the particular curve coincide with the experimental values for long columns, and so to choose the values of f as to make the plotted curve cut the experimental curve of the material under consideration at a value of l/ρ from 40 to 50; when this is done it will be found that the value of f, except in the case of cast iron, is roughly about the breaking tensile strength of the material. It will be noted that the value of f admits of being varied considerably without a corresponding great variation in the calculated strength of columns of medium length.

For the derived formula given in equation 13 the value of the constants to be used differ from those given for the Rankine-Gordon formula. Young's modulus of elasticity E remains the same, but the value of f is about two-thirds of the f of the Rankine-Gordon formula. The reason for this is that, as the hypothesis of constant stress is more clearly approximated to, the value of f to be adopted must be more nearly within the elastic limits of the material.

In the two curves, numbered 5 and 6, shown on the diagram, the value of f has been taken at the value of the strength of the material at the yield point, being for the cast tool steel 100,000 pounds per square inch, and for the annealed mild steel 40,000 pounds per square inch. It will be noticed that the curves of the derived formula intersect the curves of the Rankine-Gordon formula, and then the curves continue closely to approach each other.

One advantage of the derived formula is that three points on the curve can be fixed with a fair degree of accuracy from the experiments: thus when $l/\rho = 0$ the value of $p=f$. Now if f is taken as the strength of the material near the yield point, the value of p at the upper limit of the curve is fixed; again, when l/ρ is large the value of p becomes equal to its Eulerian value, and a second point is thus obtained which fixes the lower limit of the curve.

If, now, a column is tested having a value of l/ρ about 80 to 100, the load which will just bend the column can be determined with a fair degree of accuracy, and an experimental value is obtained for a third point on the curve. By giving a suitable value to the constant for f in the second term of the derived formula the curve can be made to coincide with the experimental value, and a curve is obtained which must approximate closely in its shape to that of the true curve.

In plotting the curves of the derived formula on the diagram the value of f adopted has been the same for all the terms containing it in the formula, and equal to two-thirds the value of the f adopted for cast tool steel and wrought iron in the Rankine-Gordon formula.

Whether the derived formula or the Rankine-Gordon formula be used in designing a column, the areas of the cross-sections obtained are nearly the same in both cases, because $P = pA = p_0 A_0$. Now, p and p_0 are the calculated values of the load per unit of area on the column by the two formulas, and as will be seen from the diagram these are nearly equal to each other, and, therefore, A is nearly equal to A_0.

For this reason the author is of the opinion that little advantage is to be gained in practice by using the derived formula in place of the Rankine-Gordon formula. The latter errs on the side of safety in giving smaller values of p for values of l/ρ greater than 40, and for columns less than this, which are unusual in practice, it approximates very closely, and further, the extreme simplicity of its form must always recommend it to engineers.

It is convenient for some purposes to put the Rankine-Gordon formula in the following form:—Let $P = f A_{,}$ where $A_{,}$ is the area of the cross-section required for a very short column, then equation 10 becomes

$$\frac{A_2}{A_{,}} = \frac{A - A_{,}}{A_{,}} = \frac{fl^2}{\pi^2 E \rho^2} \qquad (15)$$

Now $A_2 = A - A_{,}$ may be looked upon as the area required to strengthen the column against bending, and $A_{,}$ the area required to carry the load. Similarly for the derived formula, equation 13,

$$\frac{A_2}{A_{,}} = \frac{A - A_{,}}{A_{,}} = \frac{fl^2}{\pi^2 E \rho^2}\left(\frac{1}{1 + \frac{\pi^2 E \rho^2}{fl^2}}\right) \qquad (16)$$

From the diagram the curves of the two formulas for annealed mild steel intersect at a value of l/ρ about 60, and, therefore, the areas required for the cross-section of a column would be the same if calculated by either formula. The constants are for E = 30,000,000 and for $f_{,} = 60{,}000$ for the Rankine-Gordon formula, and 40,000 for the derived formula. Substituting these values (in equations 15 and 16), and putting $l/\rho = \sqrt{3{,}750}$ then for the Rankine-Gordon formula $A_2/A_{,} = \frac{3}{4}$ and for the derived formula $A_2/A_{,} = \frac{1}{6}$, that is, of seven parts of the area of the cross-section, three parts are required to resist bending and four parts are required to carry the load by the Rankine-Gordon formula, whereas one part is required to resist bending and six parts are required to carry the load by the derived formula. For the larger values of l/ρ the difference rapidly becomes less, but in all cases beyond this value of l/ρ the Rankine-Gordon formula errs on the safe side in giving the larger proportions of the area of the cross-section to resist bending. The constant c given in the table for columns with fixed ends has been derived by multiplying the constants given for the different materials for columns with round ends by ¼.

Experimentally the testing of columns with fixed ends is a somewhat difficult matter, for the few experiments that the author has been able to carry out on small columns with fixed ends, the values determined for the breaking loads agree fairly well with the constants as given. It is to be remembered, however, that columns in practice are seldom more than partially fixed, and it is better to use the greater rather than the lesser values of the constants when any doubt exists, or to modify them to suit the circumstances of the case under consideration.

ROUND-ENDED COLUMNS OF DIFFERENT SHAPES OF CROSS-SECTION.

The experimental curves and formulas already given only apply to columns of solid cross-section. When, however, the columns are of different shapes or figures of the cross-sections, such as the hollow circle or square, cross or H form, &c., the phenomena of secondary flexure influences their strength, and the formulas require modification in this respect.

A copy of a paper on "The Economic Design of Columns" is given at the end of this book, in which the influence of secondary flexure on the strength of hollow columns of circular cross-section has been considered both experimentally and analytically. It will be assumed that the reader has perused the paper at this stage, and only a brief *résumé* of the experiments and of the proposed modified formula will now be given, and then some further experimental work bearing on this subject will be referred to.

If a short hollow metal tube be tested under direct thrust in the testing-machine it will fail by secondary flexure, or wrinkling, and the tube breaks up into a series of waves. It is to be noted, however, that in every column, even with small loads, the wave formation exists; but until the elastic limit is passed, the deformation is so small as to make its observation impossible. From a comparison of the results obtained from a large number of tests it was shown that as the tube gets larger in diameter and the thickness smaller, the load producing failure becomes smaller, and it is not until the length of the tube becomes less than the length of one wave that the load producing failure approaches the resistance to compression of the material. Hence the true strength to compression of the tube is the load which produces the wave formation. From the experiments, and also from the analysis, of these waves it was found that the wave-length varied as the square root of the area of the cross-section. This result leads to the following equation for the limiting load:—

$$f = \frac{F}{1 + k\frac{\rho}{t}} \qquad (17)$$

where

f = the limiting load in pounds per square inch on a column of one wave length.

F = the strength to compression of the material in pounds per square inch.

k = a constant = $\frac{1}{8}$ for mild steel = $\frac{1}{10}$ for wrought iron.

ρ = the radius of gyration of the circular cross-section of the column about a diameter.

t = the thickness of the circular cross-section.

For circular sections, which approximate to a solid bar, the value $\left(\frac{\rho}{t} - 0.5\right)$ should be used for $\frac{\rho}{t}$; for the usual sections in practice the 0.5 is negligible.

The above formula with the constants as given applies to circular sections. Further experiments on square, triangular, and other symmetrical figures show that the form of the formula is correct, the coefficient k having a particular value for each figure, and the values so far obtained indicate that the value of k is always greater than for the circular sections. A great deal of experimental work yet remains to be done before precise values for the different figures can be given to this coefficient.

The foregoing experiments show that in estimating the strength of the column it is not sufficient only to consider the length and the radius of gyration of the cross-section, but that some provision must be made to allow for secondary flexure. A means of doing this will now be considered.

The Rankine-Gordon formula for columns is:—

$$p = \frac{P}{A} = \frac{f}{1 + c\left(\frac{l}{\rho}\right)^2},$$

where

$f =$ about two-thirds the compressive strength of the material; for wrought iron $= 36,000$ lbs. per square inch.

$A =$ area of the cross-section, in square inches.

$c = \frac{1}{9000}$ for columns with round ends.

$c = \frac{1}{36000}$ for columns with fixed ends.

$l =$ length of columns in inches.

$\rho =$ radius of gyration, in inches.

$P =$ total load on column, in pounds.

$p =$ load per square inch on columns, in pounds.

From the inspection of this formula it will be noted that the following conditions are supposed to hold true, that the strength of the column is proportional to its sectional area, the other terms being supposed constant; also that any arbitrary value of ρ can be assumed, and therefore any ratio of ρ/t, without affecting the strength of the column. It will be evident from the investigation on the effect of secondary flexure that these conditions do not hold true, and therefore the formula requires modification in this respect.

Now it has been shown from theory that the formula can be put in the following form:—

$$p = \frac{\mathrm{P}}{\mathrm{A}} = \frac{f}{1 + \frac{m\,f}{\pi^2\,\mathrm{E}}\left(\frac{l}{\rho}\right)^2} \qquad (18)$$

where E = Young's modulus of elasticity, and m a constant, the other terms having the same significations as already given.

Referring back now to equation 17, the value of f has been shown to depend on the deformation due to secondary flexure, and not upon the strength to compression of the material. The arbitrary constant f in the Rankine-Gordon formula thus becomes a variable, and, on substitution of the value obtained for f in equation 17, the following modified formula is obtained:—

$$p = \frac{\mathrm{P}}{\mathrm{A}} = \frac{\mathrm{F}}{1 + k\frac{\rho}{t} + c\left(\frac{l}{\rho}\right)^2} \qquad (19)$$

where $c = \frac{m\,\mathrm{F}}{\pi^2\,\mathrm{E}}$. This formula takes into consideration both primary and secondary flexure, and its solution involves the length and the figure and thickness of the cross-section of the column. Also, by giving values to k pertaining to the different figures of the cross-section, it can be applied generally to all forms of struts.

To determine the proportions of the economic column it is necessary to express the thickness t in terms of ρ and the area A of the cross-section. Now the area admits of being expressed in terms of its mean length and thickness. Thus, in the case of the circular cross-section, if $r =$ the mean radius, then $2\ \pi\ r\ t = \mathrm{A}$, also for thin sections $r = \sqrt{2}\,\rho$ nearly; hence, $\frac{\mathrm{A}}{2\,\pi\,\sqrt{2}\,\rho} = t$. On substitution of this value in equation 19, and differentiating with regard to p and ρ as variables, the condition obtained for a maximum is

$$\frac{\rho}{t}\,\frac{\rho^2}{l^2} = \frac{c}{k}$$

This result gives a rapid means of approximating to the proportions of the economic column, and with different values of k applies to all forms of cross-section.

From a careful comparison of the published tests on solid columns, the best value to adopt for m is unity, giving $c = \frac{\mathrm{F}}{\pi^2\,\mathrm{E}}$; and if for mild steel, $\mathrm{F} = 80{,}000$ and $\mathrm{E} = 30{,}000{,}000$, the value obtained for c is $\frac{1}{4000}$ nearly; also from the experiments on secondary flexure, the deduced value of k for mild steel is $\frac{1}{8}$, and on substitution for the constants the modified formula for mild steel columns with rounded ends is

$$p = \frac{\mathrm{P}}{\mathrm{A}} = \frac{80{,}000}{1 + \frac{1}{8}\frac{\rho}{t} + \frac{1}{4000}\left(\frac{l}{\rho}\right)^2} \qquad (20)$$

Before proceeding to discuss the relative value of the different forms of cross-section and the most economic proportions for columns, an example may be given of the use of the formula which will show the necessity of providing against the deformation due to secondary flexure.

Two circular columns, shown in Fig 4, were cut from the same length of tubing, and were thus identical in

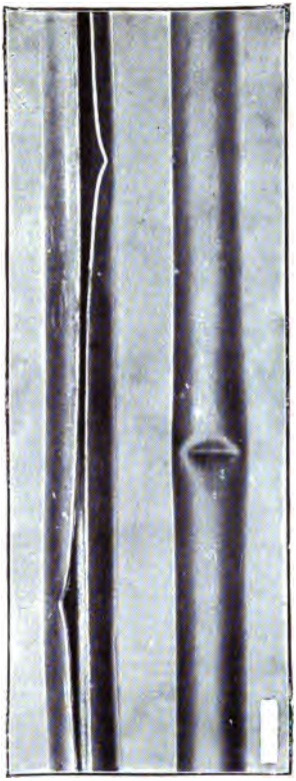

FIG 4.

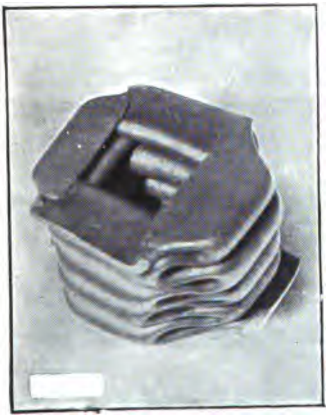

FIG. 6.

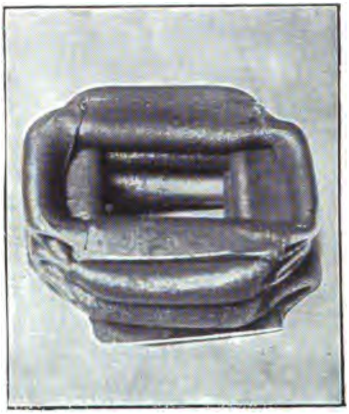

FIG. 7.

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every respect; one of these was cut along its length, and the other was left intact; the columns were then fitted with round ends and tested. The load producing failure on the cut tube was 25 per cent. less than that carried by the other.

Now if the load carried by these two columns had been calculated by the Rankine-Gordon formula, or any other of the usually known formulæ for columns, the values obtained in both cases for the load producing failure would have been the same—that is, the formulæ only take into consideration the length and the radius of gyration, and do not consider the way in which the figure or shape of the cross-section may be built up. From the inspection of the columns shown in Fig. 4 it will be seen that, owing to the secondary flexure, the manner of their failure is different, and the two sections cannot be compared together as having equal strength.

The modified formula makes provision for this, and in comparing two such columns different values for k would be used, for the circular section k is $\frac{1}{8}$, and for the split circular section of value of k is about $\frac{1}{4}$. It follows, then, in designing the cross-section of a strut, that the figure or shape and the thickness must be considered together.

In the modified formula it has been shown that for any particular figure there is some particular value of k, and when this is given, the economic proportions for the column can immediately be determined. The question then arises as to what form the figure should take in order to design the column of maximum economy. Now F and c are constants in the formula, and it is the coefficient k which depends on the figure of the cross-section. If k then is kept as small as possible, the value of the load carried will be the greatest possible. So far as the investigation on different sections has gone, the value of k is least for the hollow circular sections; hence the circular sections give the strut of maximum economy.

The foregoing demonstration proves the necessity of allowing for the influence of secondary flexure on the figure of the cross-section in determining the strength of the strut. In order to do this, it is necessary that the values of the coefficient k for the different figures of cross-section should be determined. So far this has only been done for a few figures, and the values obtained have been derived from tests carried out on small specimens, for which reason they can only be looked upon as being approximate.

The different figures of the cross-section referred to are shown on Fig. 5, and may be considered in each case as being derived from the solid bar; the thickness of the cross-section is supposed to remain uniform as its figure gradually gets larger.

From the analysis the coefficient k is of the form $\frac{\mathrm{N\,F}}{\mathrm{E}}$, where N represents what may be called the tabular number of the figure. The numbers for each figure are shown on the left-hand side of the sections, and by multiplying these numbers by the strength to compression of the material divided by its modulus of elasticity, the value of k for that figure is immediately determined.

It will be noticed that in determining the constants c and k that the factor $\frac{\mathrm{F}}{\mathrm{E}}$ is common to both; hence it follows that the economic proportions of the column do not depend upon F or E.

The illustrations in Figs. 6 to 12 show the wave formation due to secondary flexure of the different sections in Fig. 5, and will prove of interest. As already remarked, wave formation of some kind exists in every column, but within the elastic limits the smallness of the deformation makes its observation impossible.

Now consider the manner of failure of a column in which the figure of the cross-section is a square, and in which two parallel sides are thick compared with the other two sides, as shown in Fig. 13. If a short length of this column be taken and placed under a direct thrust, it will, owing to the secondary flexure, take up a wave formation similar to that shown in Fig. 14 and

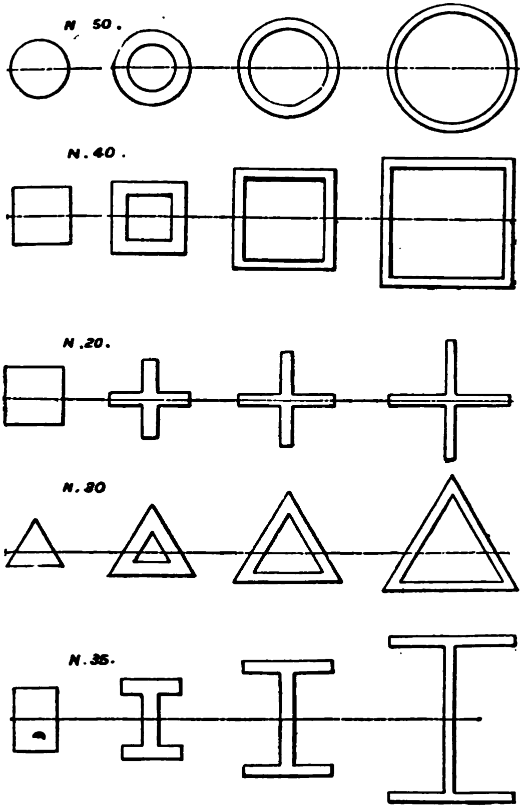


Fig. 5.

that of the specimen shown in Fig. 7; and if the sides are of equal thickness, the wave-length becomes a minimum for that particular cross-section.

Now suppose that one pair of the parallel sides becomes thin compared with the other two sides, then the wave-length increases, and in the limit when the sides disappear

Fig. 13.

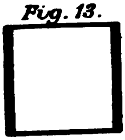


Fig. 14.

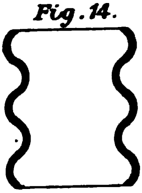


Fig. 15.

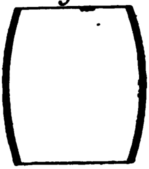
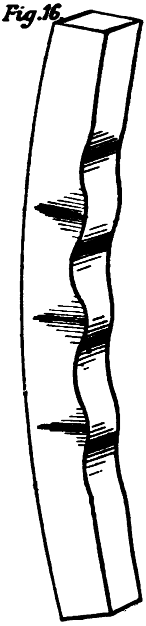


Fig. 16.



altogether, the two thick sides become two separate columns, as shown in Fig 15. The strength, then, of a column of this kind to resist secondary flexure is a maximum when the sides are of equal thickness, and the manner of its failure is somewhat as shown in Fig. 16. The wrinkling or crumpling up of the side of the column introduces complications, and the laws of simple bending can no longer be said to hold true, more especially when the elastic limit is passed; also, the thinner one pair of

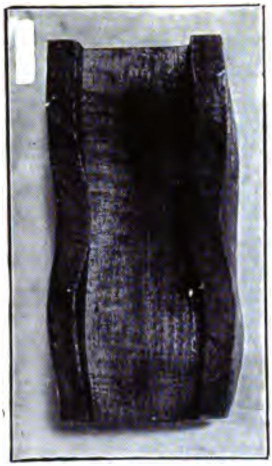


Fig. 8.

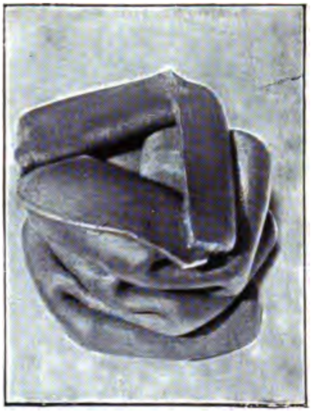


Fig. 9.

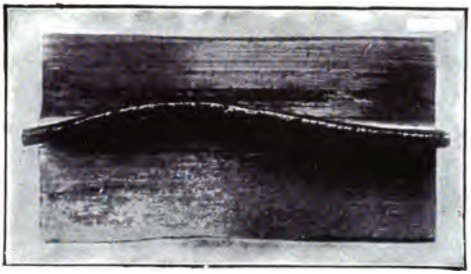


Fig. 10.

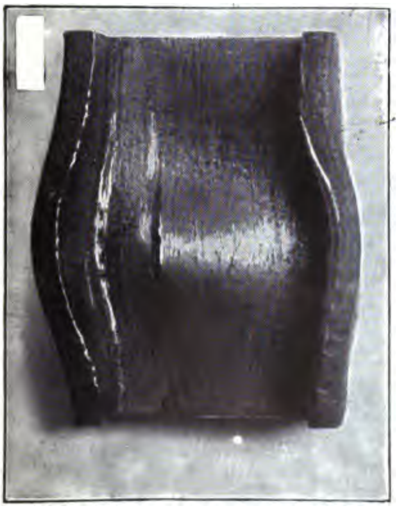


Fig. 11.

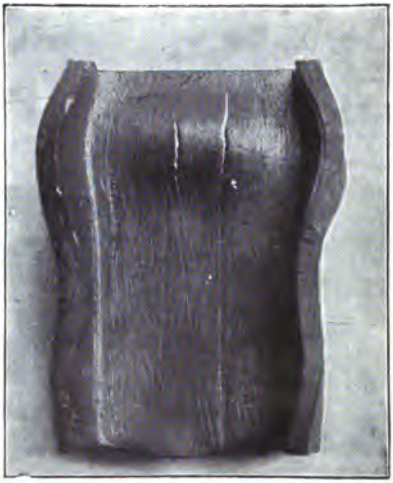


Fig. 12.

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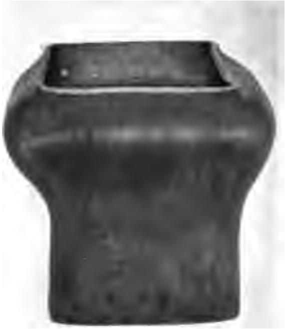

Fig. 17.

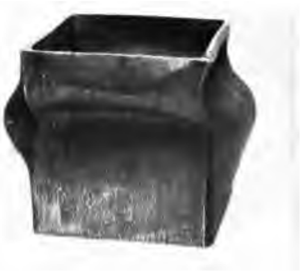

Fig. 18.

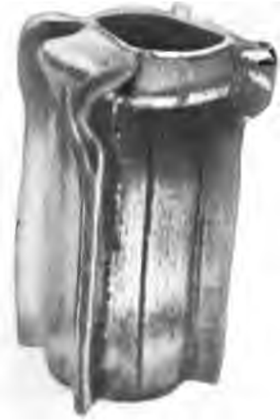

Fig. 22.

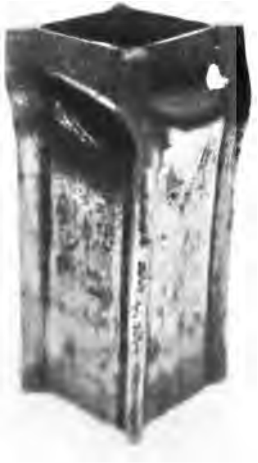

Fig. 23.

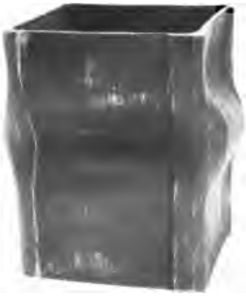

Fig. 24.

To follow Fig. 12.]

sides becomes relatively to the other the more pronounced does this effect become. The weakness of the thin sides in bracing or connecting together the thick sides for the transmission of the bending and shear stresses is evident.

Figs. 17 and 18 show the effect of secondary flexure on two different sections in which the sides were of equal and unequal thicknesses respectively.

It will be noted in the case of the square section with sides of unequal thickness that the wave formation causes the section to appear rectangular in plan, and in the case of a circle, in which two sides on one diameter are thin compared with two sides on a diameter at right angles to the first, and in which the thickness varies point by point, then the wave formation in plan appears approximately as an ellipse.

The values of k for the different sections already given only apply to sections of a uniform thickness: for sections in which the thicknesses of the sides are unequal, the values of k multiplied by $\sqrt{n}$ where n is the ratio of the thicknesses of the sides, can be used. The reasons for adopting this coefficient are given in the Appendix.

It has been shown that for a hollow circular column there is a particular diameter and thickness of the cross-section which will give the most economical result. The question then arises as to the effect of stiffening such cross-sections or making them of a cellular form, as shown in Figs. 19 and 20, and under what conditions will they be stronger than the plain hollow cross-section of a similar figure.

On Fig. 21 is shown a sketch of the manner of failure of a column of the circular cellular type, the length of which depends upon the proportions of the cells to the diameter; and superposed on this wave are smaller waves, the lengths of which depend upon the thickness of the cells compared with the figure of the cross-section thereof.

Some different types of stiffened cross-sections are shown on Figs. 22 to 24, which will give some idea of the manner of failure of such sections and of the kind of wave formation developed.

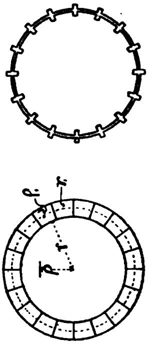

FIG. 19.

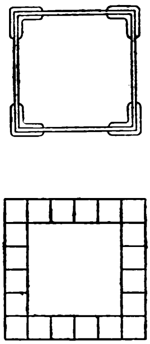

FIG. 20.

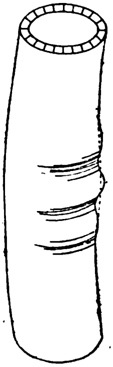

FIG. 21.

The analysis given in the Appendix shows that for such stiffened cross-sections the modified formula is of the form:—

$$p = \frac{F}{1 + u\frac{\rho_{,}}{t} + k\frac{\rho}{\rho_{,}} + c\frac{l^2}{\rho}} \qquad (21)$$

Where ρ and $\rho_{,}$ are the radii of gyration of the cross-section and of a cell respectively, u, k, and c are constants, of these c remains as before. The experiments so far carried out have not been sufficient to determine u and k. A great deal of work yet requires to be done before precise values can be given to these constants. The formula, however, shows that:—

$$\frac{\lambda_{,}}{t} \propto \frac{\lambda}{\rho} \propto \frac{l}{\rho}$$

Where $\lambda_{,}$ and λ are the wave-lengths and it is possible to approximate to these and thus obtain roughly what the economic proportions should be.

APPENDIX.

THE DERIVED RANKINE-GORDON FORMULA.

THE manner of deriving the formulæ given in equations 11 and 12 shows that by continued substitution for p and $p_{,}$ the equations can be expressed in the form of a series. Thus—

$$p=\frac{f}{1+\frac{fl^2}{\pi^2E\rho^2}\left(\frac{1}{1+\frac{\pi^2E\rho^2}{fl^2}+\left(\frac{\pi^2E\rho^2}{fl^2}\right)^2\cdots+\left(\frac{\pi^2E\rho^2}{l^2}\right)^n\frac{1}{f^{n-1}(f-p)}}\right)}$$

and (22)

$$p_{,}=\frac{f}{1+\frac{\pi^2E\rho^2}{fl^2}+\left(\frac{\pi^2E\rho^2}{fl^2}\right)^2\cdots\left(\frac{\pi^2E\rho^2}{l^2}\right)^n\frac{1}{f^{n-1}(f-p)}} \qquad (23)$$

Let $a=\frac{\pi^2E\rho}{fl^2}$ then summing up as a geometric series and putting p equal to o in the last term of the series

$$\frac{p_{,}}{f}=\frac{a-1}{a^n-1} \quad \text{and} \quad \frac{p}{f}=\frac{a(a^{n-1}-1)}{a^n-1} \qquad (24)$$

Now a is a variable, and when its value approaches 1 the limit gives $\frac{p}{f}=\frac{n-1}{n}$ and $\frac{p_{,}}{f}=\frac{1}{n}$. Thus, for the particular value of $\frac{\pi^2E\rho^2}{fl^2}=1$, the ratio of p to f or $p_{,}$ to f depends upon the number of terms that are taken in the series.

In equations 22 and 23 if the Eulerian value is substituted for p the last term is $-\frac{a^n}{a-1}$, the formulæ then return to the form

$$\frac{p_{,}}{f}=1-a \text{ and } \frac{p}{f}=a \text{ or } p=\frac{\pi^2E\rho^2}{l^2}.$$

Dividing equation 22 by 23 the value of equation 3 becomes

$$\frac{\delta y}{\rho^2}=\frac{p_{,}}{p}=\frac{a-1}{a(a^{n-1}-1)}.$$

On Fig. 25 a number of curves have been plotted to equation 24, different values being given to n and a. These curves show

C

at a glance the effect of taking a large value for *n*. With *n* large a part of the curve is nearly a straight line until it approaches the Eulerian curve, and after that it nearly coincides with it,

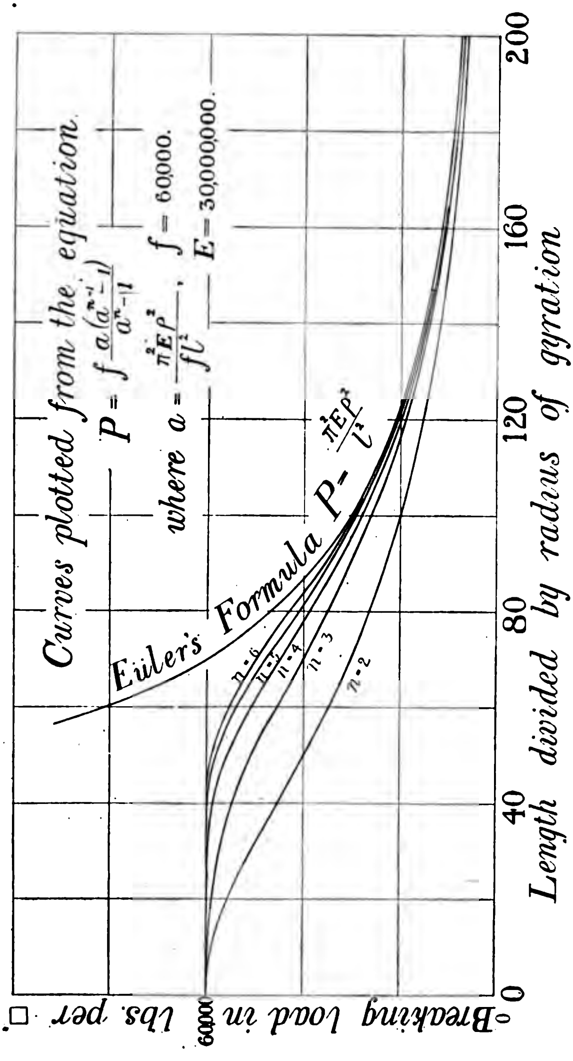

FIG. 25.

if $n = 2$ the Rankine-Gordon curve is obtained, and if $n = 3$ the modified formula as given in the text.

COLUMNS OF SQUARE OR H CROSS-SECTION AND SIDES OF UNEQUAL THICKNESS.

The value of the constant k, as given in the text, only applies to cross-sections of a uniform thickness. When the cross-sections are not of a uniform thickness the determination of the correction required in the constant k can, for square or H cross-sections, be approximately determined by using the analysis given on pages 20–23 of the paper on the "Economic Design of Columns" as follows:—

Consider a column of H section which, under the applied load, breaks up into a wave-length, as shown on Fig. 26. Using the notation on the figure, and assuming the curve of the flange to be sinusoidal, then the intensity of the stress p at any section x of the web is—

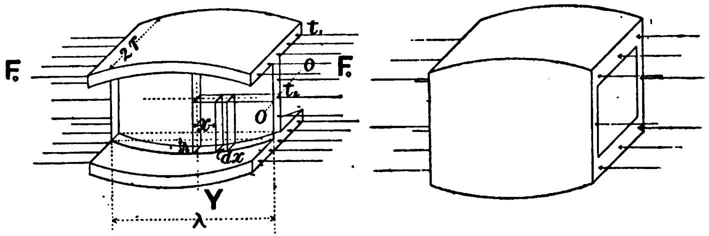

FIG. 26.

$$p = Ee = E\frac{h}{r}\cos\frac{x}{k}$$

where e = the strain, and

$$S = \frac{Eht_2}{r}\int_0^{\lambda/2}\cos\frac{x}{k}dx$$

where S = one-half of the total pulls in the web, and

$$M = \frac{Eht_2}{r}\int_0^{\lambda/2}\int_0^{\lambda/2}\cos\frac{x}{k}dx = \frac{Eht_2\lambda^2}{r\pi^2}$$

where M is the moment of the pulls round the centre line Y. This is equal to F_1h, hence $F_1 = \frac{Et_2\lambda^2}{r\pi^2}$. The other part of the load carried by the flange is $F_2 = \frac{\pi^2EI}{\lambda^2}$ then

$$F_0 = F_1 + F_2 = E\left(\frac{t_2\lambda^2}{r\pi^2} + \frac{\pi^2I}{\lambda^2}\right) \quad \text{and} \quad I = \frac{2rt_1^3}{12},$$

the depth of the web being assumed in this case of equal breadth to the flanges. Substituting

$$F_0 = E\left(\frac{t_2\lambda^2}{r\pi^2} + \frac{\pi^2 r t_{,}^3}{6\lambda^2}\right)$$

on differentiation with regard to λ and equating for a minimum

$$\frac{\delta F_0}{\delta\lambda} = 2E\left(\frac{\lambda t^2}{\pi^2 r} - \frac{\pi^2 r t_{,}^3}{6\lambda^3}\right) = O, \quad \text{giving } \lambda^2 = \frac{\pi^2 r t_{,}}{\sqrt{6}}\sqrt{\frac{t_{,}}{t_2}},$$

and on substitution

$$F_0 = \frac{2E}{\sqrt{6}} t_{,}\sqrt{t_{,}t_2}$$

let n be the ratio of the thickness of the flange to the web—that is, $nt_2 = t_{,}$—then

$$F_0 = \frac{2E}{\sqrt{6}} \; \frac{t_{,}^2}{\sqrt{n}}$$

the area of one flange is $A = 2rt_{,}$ giving

$$f = \frac{F_0}{A_0} = \frac{E}{\sqrt{6}} \; \frac{t_{,}}{r\sqrt{n}}$$

Where F_0 is the total load on one flange, and f the intensity of the stress on the flange, another portion of the applied load is carried by the web. This is small when t_2 is much less than $t_{,}$. The wave formation in the flanges forces the web to take up a wave formation of the same length. At the same time there are smaller waves superposed on the first wave. To allow for the load in terms of these smaller waves is difficult, and it is better, as a first approximation, to assume the intensity of the stress on the web portion to be the same as that on the flanges. The error will be small in making this assumption, because the area of the cross-section of the web would usually be small compared with that of the flanges. Let, then, A refer to the whole area of the cross-section and F to the total applied load, then

$$f = \frac{F_0}{A_0} = \frac{F}{A} = \frac{E}{\sqrt{6}} \; \frac{t_{,}}{r\sqrt{n}}$$

and on substitution in the modified formula in a similar manner to that given for the circular sections it becomes

$$p = \frac{P}{A} = \frac{f}{1 + \sqrt{n}\,k\frac{\rho}{t_{,}} + c\frac{l^2}{\rho^2}}$$

Where $t_{,}$ is the thickness of the thicker side of the cross-section when $n = 1$ the formula returns to its original form, the constants already given for k and c apply. As already remarked, the experiments being carried out on small sections, the values obtained are more or less approximate. This investigation, however, gives a means of allowing for non-uniform sections, and by analogy can be used for the values of k given for other sections.

Columns of Stiffened or Cellular Cross-section.

Referring back to Fig. 21, the effect of secondary flexure on a column of the circular cellular type causes a long wave to be developed, the length of which depends upon the proportions of the cells to the diameter of the cross-section. Superposed on this wave are smaller waves, the lengths of which depend upon the thickness of the cells compared with the figure of their cross-section. It is required to determine the proportions between the depth of the cell and its thickness and the diameter of the cross-section which will give the most economical result. The following solution indicates roughly what these proportions should be for a particular type of cross-section.

Let the area of the ring-sections shown on Fig. 19 be $A_{,} = 4\pi rt$ where t is the thickness of the inner or outer ring, if A = the total area of the cross-section, then $A_{,} = mA$ where m is a constant depending on the type of cross-section; also let ρ and $\rho_{,}$ be the radii of gyration of the cross-section and of the cell respectively.

On page 20 of the paper on the "Economic Design of Columns" the following equation is given:—

$$F = F_{,} + F_2 = \frac{2Et\lambda_2}{\pi r} + \frac{\pi^2 EI}{\lambda^2}$$

and on substitution of the above values

$$F = EA\left(\frac{m\lambda^2}{\pi^2 r} + \frac{\pi^2\rho^2}{\lambda^2}\right)$$

On differentiation and equation for a minimum this gives

$$\lambda^2 = \frac{\pi^2 r\rho_{,}}{\sqrt{m}}$$

and

$$f = \frac{F}{A} = \frac{\sqrt{m}\,E\rho_{,}}{a\rho}$$

Where $a = \frac{r}{\rho}$ = a constant depending on the figure or shape of the cross-section.

Assume now that $f_{,}$ is the strength to compression per unit of area of the cellular cross-section—that is, neglect for the present the effect of the smaller waves upon the strength of the cell.

Then, substituting in a similar manner to that already given in the analysis for the circular sections, the modified formula takes the form—

$$p = \frac{f_{,}}{1 + \frac{f_{,}}{E}\left(\frac{\sqrt{m}\,\rho}{a\,\rho_{,}} + \frac{l^2}{\pi^2\rho^2}\right)}$$

Now consider one cell of the cross-section. If there were no tension in the circular rings due to the longer wave, its resistance to compression would be of the same form as that already given, viz. :—

$$f_{,} = \frac{F}{1 + \frac{bF}{E}\frac{\rho}{t}}$$

where b is a constant depending on the figure or shape of the cross-section. The tension in the rings due to the longer waves has the effect of altering the length of these waves slightly. This will be evident from the analysis given herewith in the paper on the "Web Stresses in Plate Girders." As a first approximation, the effect of this tension may be considered negligible.

Substituting in the previous equation for $f_{,}$ the formula becomes

$$p = \frac{F}{1 + \frac{F}{E}\left(\frac{b\rho_{,}}{t} + \frac{\sqrt{m}\,\rho}{a\,\rho_{,}} + \frac{l^2}{\pi^2\rho^2}\right)}$$

$$= \frac{F}{1 + u\,\frac{\rho_{,}}{t} + k\frac{\rho}{\rho_{,}} + c\,\frac{l^2}{\rho^2}}$$

or, since $\lambda_{,}^2 \,\alpha\, \rho_{,}\, t$ and $\lambda^2 \,\alpha\, \rho\, \rho_{,}$

$$p = \frac{F}{1 + U\frac{\lambda_{,}^2}{t^2} + K\frac{\lambda^2}{\rho_{,}^2} + G\frac{l^2}{\rho^2}}$$

Now, in cellular or similar cross-sections, if the diameter is varied, the size of the cells will vary in the same proportion, and for a given figure or shape the ratio of ρ to $\rho_{,}$ is nearly constant. Let, then, $\rho/\rho_{,} = g$ and $A = e\rho_{,}t = \frac{e}{g}\rho t$. Substituting these values the formula obtained is

$$p = \frac{F}{1 + \frac{ue\rho^2}{g^2A} + kg + c\frac{l^2}{\rho^2}}$$

Let p and ρ be the only variables in this equation, then, on differentiation and equating to zero, the condition obtained for a maximum is

$$\rho^2 = \sqrt{\frac{c}{ue}}\, g \sqrt{Al} \text{ or } \frac{\rho_{\prime}}{t} \frac{\rho^2}{l^2} = \frac{c}{u}$$

and

$$p = \frac{F}{1 + kg + \dfrac{2\sqrt{uec}\, l}{g\sqrt{A}}}$$

If, on the other hand, p, ρ and $\rho_{\prime}$ are taken as the variables, and the other terms assumed to remain constant, then, on differentiation and equating to zero, the condition obtained for a maximum is

$$u\frac{\rho_{\prime}}{t} = \frac{k}{2}\frac{\rho}{\rho_{\prime}} = c\frac{l^2}{\rho^2}$$

or

$$\frac{\lambda_{\prime}}{t} \propto \frac{\lambda}{\rho_{\prime}} \propto \frac{l}{\rho}$$

and

$$p = \frac{F}{1 + 2\dfrac{\sqrt[4]{uec}\,\sqrt{2kl}}{\sqrt[4]{A}}}$$

The difficulty in using the results just obtained is in the determination of the constants u and k. The experimental work so far carried out has not been sufficient to justify giving them even approximate values.

The analysis, however, indicates the proportions that ought to be observed, and, given approximate values of $\lambda_{\prime}$ and λ, the other proportions can be determined.

BRACED COLUMNS.

It has been shown, when considering columns of square or H cross-section with the sides of unequal thickness, that a correction is required in the term of the modified formula which allows for secondary flexure. A further question arises, when dealing with such sections, as to the minimum thickness of the web compared with that of the flanges.

The case of the H cross-section shown on Fig. 26 will be first considered, and then columns of the type shown on Fig. 27, in which the separate columns are connected together by means of webs or bracing.

From the views that have been put forward when dealing with solid columns it has been assumed that the load which produces failure in a column will bend it to its proof deflection; also that the curve into which the column bends is a curve of sines.

Now, if in a beam or column there is a variation of the bending moments, then shearing stresses are set up in the bracing or web. Usually in a beam or girder it is the shearing stresses which give rise to the bending moments in the flanges, whereas in columns it is the variation of the bending moments which give rise to the shearing stresses. If, then, a column is bent to its proof deflection, then so also should the web, if properly designed, come to its proof stress.

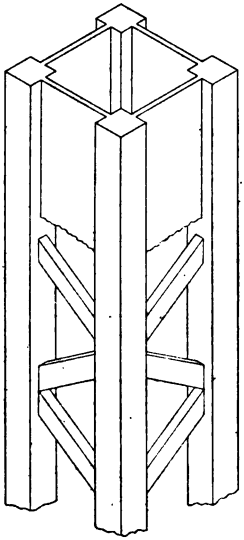

Fig. 27.

Now consider a rolled beam of H section supported at its extremities and bent to a sinusoidal curve. If w is the intensity of the load per unit of area and δ the deflection, the bending moment

$$M = \frac{w\,\delta\,l^2}{\pi^2} \quad \text{and} \quad S = \frac{w\,\delta\,l}{\pi}$$

where S is the shearing force at one support. Hence

$$S\frac{l}{\pi} = M = \frac{fI}{y}$$

from which, if f the proof strength is known, the shear stress can be determined.

Now consider the rolled beam loaded, as in a long column, and in which bending takes place round the neutral axis OO, then Euler's analysis gives

$$S\frac{l}{\pi}=\frac{\pi^2EI}{l^2}\cdot\frac{fl^2}{\pi^2Ey}=\frac{fI}{y}$$

that is, S depends upon the value taken for f, and varies inversely as the length; also, S has the same value as in the corresponding case of a beam. This is owing to Euler's analysis only considering bending and not allowing for the area required to transmit the load.

The Rankine-Gordon formula makes provision for this, and gives the first approximation. Thus

$$P=\frac{fA}{1+\frac{f}{\pi^2E}\frac{l^2}{\rho^2}}\quad\text{and }\delta=\frac{fl^2}{\pi^2Ey}$$

giving—

$$S\frac{l}{\pi}=P\delta=\frac{fA}{1+\frac{f}{\pi^2E}\frac{l^2}{\rho^2}}\cdot\frac{fl^2}{\pi^2Ey}$$

and

$$S=\frac{fI}{y}\,\frac{1}{\frac{l}{\pi}+\frac{\pi^2E\rho^2}{fl}}$$

The value of S obviously admits of a maximum value, and on differentiation, with regard to S and l as variables, the condition obtained for a maximum is

$$l^2=\frac{\pi^2E\rho^2}{f}$$

and on substitution

$$S_{max}=\frac{I}{y}\frac{f^{3/2}}{2\sqrt{E\rho^2}}=\frac{A\rho}{y}\frac{f^{3/2}}{2\sqrt{E}}$$

or the shear stress is one-half the value that would be given by Euler's analysis for this particular value of l.

A similar analysis for the derived formula gives

$$S_{max}=\frac{A\rho}{y}\frac{f^{3/2}}{\sqrt{E}}\frac{1}{2.5}\text{ nearly.}$$

The relation just obtained, showing that there is a particular value of the length of a column in which the value of S is a maximum, seems now to be pointed out for the first time. When the relation given in equation 15 is remembered its truth will be evident.

The value of A relates to the whole area of the cross-section. Let B be the area of the cross-section of the web, then $A = mB$ and

$$S = \frac{B\rho}{y} \; \frac{m f^{3/2}}{2\sqrt{E}}$$

The effect of secondary flexure on the value of f has been neglected. The result of doing so is somewhat to increase the area of the web—an error on the side of safety. In using the formula it is to be noted that it only gives the minimum area of the web, and the value obtained must be considerably increased. The web itself requires stiffening, and a formula applicable for this purpose is given in the paper on "Web Stresses in Plate Girders" herewith; also, it must be strong enough to take up the twisting moments to prevent distortion taking place sideways. The formula being derived from the Rankine-Gordon formula, a high value for f must be taken—that is, about 60,000 to 80,000 pounds per square inch. If this is done it will be found that the area of the web will be about one-fifth to one-tenth the total area of the cross-section.

In the case of the Quebec Bridge, the area of the cross-section of the bracing to that of the whole area of the cross-section of the lower chord member was about one-seventy-fifth, and, judged from the foregoing analysis, about one-eighth of what was necessary to develop its full strength.

The subject may be approached from another point of view. In rolled beams the area of the cross-section of the web is approximately equal to the area of one flange of the cross-section, or about one-third the total area. Now, in beams of this kind, the proportion of the depth to the span is about one-tenth to one-twelfth, giving a value of l/ρ about 25. Suppose now the beam were lengthened so that l/ρ was from 50 to 75, then, as the shearing stress varies inversely as the length, the area of the web would be from one-half to one-third its previous value—that is, about one-fifth to one-seventh of the area of the cross-section. Now, for a column of the critical length just considered, l/ρ is about 60 to 65, and the area of the cross-section to transmit bending then equals the area to transmit the load; consequently, compared with the beam, the area of the web would be about one-tenth the total area of the cross-section. For reasons already pointed out the probable value would be about one-half of this.

It follows then for columns usual in practice—that is, with values of l/ρ, about 60 to 80—that, if the web or bracing is designed as it would be for a beam of one-half the area of the cross-section of the column, it will be sufficient to develop the full strength of the column. The transition from columns of the H section to columns of the type shown on Fig. 26 will now present no difficulty, as the web or bracing connecting the separate columns together must be similar in its proportions to that of the H section. If bracing is adopted the area of its cross-section is given by the equation $a = \frac{S}{f} Sec\ \theta$ where θ is the angle of inclination of the bars to the horizontal.

Further, it is to be noted that diaphragm plates or cross bracing would be required at intervals, more especially near the middle of the column, to prevent change of the figure of the cross-section.

THE

ECONOMIC DESIGN OF COLUMNS.

THE proportions adopted in the design of beams and columns for engineering structures are to a large extent empirical. This is inevitable when it is considered that in many cases problems are propounded which in the present state of our theoretical knowledge no satisfactory solution can be given, and also that in some cases, although a solution based upon theoretical considerations may be obtained, yet owing to the want of experimental data by which it may be confirmed, no practical use can be made of it. The designer under such conditions must fall back on his own and other experience to determine the proportions which appear to be most suitable under the circumstances, and it often happens that these proportions sanctioned by experience, and which may be looked upon as the outcome of a long course of trial and error, prove to be the proportions indicated by theoretical reasoning as the analysis becomes more rigorous and the experimental data by which it may be confirmed more complete. The design of hollow circular columns may be cited as a case in point. It has long been known that a length of about 25 diameters and a thickness of about 1-16th the diameter would give good results. The proportions were purely empirical, and were not in any way based upon theoretical considerations. More recently, however, both theory and experiment show that the range of the best ratio of length to diameter of a column is comparatively small, and that the values adopted by experience come within this range.

The problem then that naturally presents itself is, what should be the proportions between the diameter and thickness of the column when the length and load are known in order that the material may be distributed to the best advantage. If the column is made of large diameter and small thickness it will fail by wrinkling or secondary flexure of the sides of the column, and if of small diameter and great thickness it will fail by bending or primary flexure. Hence it follows that for some particular diameter and thickness it will fail equally by secondary or primary flexure. A column in which the length, diameter and thickness are so proportioned as to obtain this result is called an economic column, and the load it will carry for a given quantity of material is a maximum.

During the past two years experiments have been carried out in the Engineering Laboratory of Trinity College on mild steel tubes of various lengths, diameters, and thicknesses with a view of determining experimentally the conditions under which failure takes place in a column either by primary or secondary flexure, and thus indirectly to obtain some definite information of the proportions of the economic columns. A detailed description of these experiments was published in the "Proceedings of the Institution of Mechanical Engineers," June, 1905, and it is not proposed in this paper to refer further to the particulars of these experiments, but rather to point out some interesting phenomena which further experiments have shown occur in the testing of columns when studied experimentally, and also to show that the analysis derived from the results of these experiments admits of being simply expressed, and leads to a modification of the formulæ in general use at the present time for the design of columns.

On Plate I. a diagram is shown which has been deduced from a large number of experiments on mild steel tubes with round ends, and shows the influence of thickness in determining the failure of the column. The various curves relate to a series of columns in which the area of the cross section is kept constant, the upper curve being for the limiting case when the column becomes a solid bar. Values of l/ρ, the length divided by the radius of gyration, are plotted on a horizontal line as base, and values of p the load in pounds per square inch on the column vertically. From an inspection of these curves the influence of the ratio of the diameter to thickness is clearly shown, the lower curves being for the case of the thinner tubing. It will be noted that instead of these curves gradually approaching to the strength of compression of the material, that the lower curves approach a definite value, and then quickly rise to the strength of compression. For the solid bar and tubes which approximate to the solid bar the curves show two well-defined humps, but for thin tubes the curves show three humps, one very abrupt, rising almost vertically to the strength of compression, and two other humps similar to those obtained for the solid bar.

It has already been remarked that the failure of a column takes place either by primary or secondary flexure. One of the principal results of these experiments has been to show that some remarkable wave phenomena occur in connection with the secondary flexure of a column.

On Plate II. the wave forms assumed by a number of short tubes of varying diameters and thicknesses are shown, the diameters and gauges of the tubes being in the order given by the following table:

Diameter	Standard Wire Gauge
$\frac{3}{8}$ inch	30–30–26–26–22–22–18
$\frac{1}{2}$,,	30–30–26–26–22–22
$\frac{3}{4}$,,	30–30–26–26–22–18
1 ,,	30–30–26–26–22–18
$1\frac{1}{4}$ inches	26–26–22–22

The short tubes were tested under a direct thrust in the testing machine, the ends being truly squared.

From a comparison of the results obtained on a large number of tubes tested, it appears that as the diameter of the tube gets larger and the thickness smaller that the limiting load on the tube of one wave-length becomes smaller.

This is clearly shown on Plate I.; the breaking stress curve with two humps is that for the solid bar, and forms the uppermost curve of the series; then, as the bar becomes of hollow circular section and of larger diameter and thinner, the humps in each case form a lower curve of the series, and it is not until the length of the tube becomes less than one wave-length that the strength to compression of the material is reached—this is the cause of the very abrupt rise of the third hump on the thin sections. The experiments show that the length of these waves varies as the square root of the area of the cross section of the tubes and the analysis given in the Appendix leads to the same conclusion. The value of the limiting load on one wave-length is given by the equation—

$$f = \frac{F}{1 + k\frac{\rho}{t}}$$

where f=the limiting load in pounds per square inch on a column of one wave-length,

F=the strength to compression of the material in pounds per square inch,

k=a constant = $\frac{1}{8}$ for mild steel,

ρ=the radius of gyration of the circular cross section of the column about a diameter,

t=the thickness of the circular cross section of the column.

The curves show that the failure of columns is a complex phenomenon, and points to three well-defined ways in which the column fails. One way is, for value of $l/\rho > 120$ for the solid bar and varying to values of $l/\rho > 240$ for the thinnest tubing, when the failure is sensibly elastic, and the stress on the material is within the elastic limit. This can be verified by removing the load from the column upon which it returns to its original straightness. A second way is that in which, for values of $l/\rho > 120$ and < 40 for the solid bar and $l/\rho < 240$ and > 80 for the thinner tubing, the load now producing failure partly stresses the material beyond the elastic limit, so that on taking off the load it will be found that the column has received a permanent set or deformation. Lastly, for values of $l/\rho < 40$ for the solid bar and of $l/\rho < 80$ for the thinner tubing, the load producing failure partly stresses the material beyond its strength of compression in the case of the solid bar, and partly causes the phenomenon of buckling or wave formation in the case of the thin tubing, so that in this third case the strength of the wave to resist compression is the true strength of compression of the column. The part played by secondary flexure on the failure of thin columns and the wave formation produced is shown on Plate III., on which are shown tubes ranging from $\frac{3}{8}$ in. to $1\frac{1}{4}$ in. diameter and of 30 S.W.G. They are selected from a number of tubes of these dimensions which had been tested; the tubes in all cases being thin show the wave formation.

On Plate IV. tubes of the same diameters are shown, but of 26 S.W.G.; in these the wave formation on the smaller sizes is not well defined, because, these tubes being thick compared with their diameter, the secondary flexure did not play so important a part in the failure of the column. For the tubes having the larger diameters, the wave formation will be seen to be well marked, being produced in this case as the tubes were thin compared with the diameters.

The curves on Plate I. relate to tests that were carried out on mild steel tubes, of which the average strength to tension was 72,000 pounds per square inch. The determination of the strength to compression was a matter of some difficulty, owing to the lengths of the tubes tested being less than one wave-length. The average of the results obtained from the experiments gave a value of 80,000 pounds per square inch, and this value was adopted in plotting the curves. Experiments were also made for the determination of Young's modulus of elasticity, giving an average value of 30,000,000 pounds per square inch. This value has been adopted throughout.

The following phenomena were observed during the carrying out of the tests:—

That for all columns there is a certain value of l/ρ beyond which the failure is sensibly due to elastic bending, the load producing failure varying inversely as the square of the length.

That as the value of ρ/t increased the value of l/ρ increased for the load producing failure by elastic bending. The curves on Plate I. show this.

When the failure of the column takes place by secondary flexure for a large range of the length there is little variation in the breaking load. Hodgkinson, in his experiments on wrought iron columns, has previously remarked on this.

As the load approached the breaking load, there was on the short lengths a decided yielding of the tube before failure.

In all cases of long columns some elastic deflection was apparent before failure; the columns were then in a critical condition, and it was not till the deflections exceeded a certain amount that collapse took place.

On Plate I. a number of curves are shown plotted to the following modified form of the Rankine-Gordon formula—

$$p=\frac{P}{A}=\frac{F}{1+k\frac{\rho}{t}+c\left(\frac{l}{\rho}\right)^2} \quad (1)$$

where p = the load per square inch on the column in pounds,
P = the load on the column in pounds,
A = the area of the cross section in square inches,
F = the compressive strength of the material in pounds per square inch, for mild steel = 80,000,
l = the length of the column in inches,
ρ = the radius of gyration of the cross section about a diameter in inches,
t = the thickness of the cross section in inches,
c = a constant, for mild steel $= \frac{1}{4{,}000}$ for columns with round ends and $\frac{1}{16{,}000}$ for fixed ends,
k = a constant, for mild steel = $\frac{1}{8}$,

and on substitution for the constants the formula for mild steel is

$$p=\frac{P}{A}=\frac{80000}{1+\frac{1}{8}\frac{\rho}{t}+\frac{1}{4000}\left(\frac{l}{\rho}\right)^2} \quad (2)$$

For the solid bar and tubes which approach the solid bar $\frac{1}{8}\left(\frac{\rho}{t}-\cdot5\right)$ should be used for $\frac{1}{8}\frac{\rho}{t}$. For ordinary sections the value of $\frac{1}{8}\frac{\rho}{t}$ is sensibly correct.

The reasons for adopting this formula in place of the one previously obtained* are given in the Appendix. It is of a simpler form, and the values obtained by its use give a much closer approximation to the results obtained from the experiments, and, more especially, having a theoretical basis, it is in every way more satisfactory. Let A be the area of the cross section, d the mean diameter of the column, then $\pi dt = A$. Also $\rho = \frac{d}{\sqrt{8}}$ approximately, hence $t = \frac{A}{8{\cdot}9\,\rho}$ substituting in equation (2)

$$p = \frac{P}{A} = \frac{80000}{1 + 1{\cdot}1\frac{\rho^2}{A} + \frac{1}{4000}\left(\frac{l}{\rho}\right)^2} \qquad (3)$$

Now let p and ρ be the only variables in this equation, then, on differentiation and equating to zero, the condition obtained for a maximum is $\rho^2 = \frac{\sqrt{A}\,l}{66}$ (4)

which is similar to that obtained previously from the experiments,* and differing only in that the coefficient is $\frac{1}{66}$ in place of $\frac{1}{94}$. This change in the value of the coefficient is caused by the curves of the Rankine-Gordon formula not coinciding with those obtained from the experiments for the economic sections (the amount of the variation being clearly shown on Plate I.). For the usual values of l/ρ in practice about 40 to 80, the value of the coefficient may be varied from $\frac{1}{66}$ to $\frac{1}{94}$, but for other values in which the Rankine-Gordon curves closely approximate to the experimental curves the value of $\frac{1}{66}$ must be adhered to. No simple formula can be expected to give values which will fit in closely with the experimental curves. These curves, as already remarked, show well-defined humps, and any mathematical expression which will give values for such curves must be of a more or less complicated form. For this reason it has been deemed advisable to keep to the more well-known form of the Rankine-Gordon formula and to give such values to the constants as will give good average values obtained from the experiments.

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* Proc. I. Mech. E. 1905.

The formula takes into consideration both the primary and secondary flexure of the column. Since the column can fail in either of the two ways it will be of maximum strength when the strength to resist primary flexure is equal to that for secondary flexure—that is, the strength of the side of the column for secondary flexure must be equal to the stress produced by primary flexure and the load. The proportions of the column to attain this can rapidly be approximated to by using equation (4) in the following form* :—

$$\left(\frac{\rho}{t}\right)\left(\frac{\rho}{l}\right)^2 = \frac{8{\cdot}9}{66^2 \text{ to } 94^2} \qquad (5)$$

Now substitute the value obtained for ρ in equation (4) in equation (3); this gives

$$p = \frac{P}{A} = \frac{80000}{1 + \frac{1}{30}\frac{l}{\sqrt{A}}} = \frac{80000}{1 + \frac{1}{2000}\left(\frac{l}{\rho}\right)^2}$$

This cubic equation determines the economic column in terms of P and l and the constants, and is identical in form with that previously obtained by the author*—the value of the coefficient $\frac{1}{2000}$ taking the place of $\frac{1}{3500}$ previously given. As already remarked, this is due to the experimental curves not coinciding with those obtained from the Rankine-Gordon formula.

It will be evident from the foregoing investigation that the experimental work yet to be carried out on columns will be to determine the relative strengths of the different sections in common use to secondary flexure, and thus obtain values for the determination of the constant k in the modified form of the Rankine-Gordon formula.

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* Proc. I. Mech. E. 1905.

The results that have been so far obtained on small irregular sections indicate that it will be greater than for the circular sections, and this is what would naturally be expected.

In conclusion, the author believes it is now possible for the first time to design a column with reasonable economy by the application of theoretical principles rather than by empirical rules based on experience.

APPENDIX.

EULER'S FORMULA FOR COLUMNS.

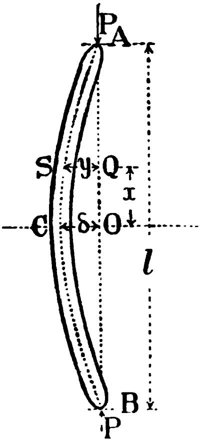

FIG. 1.

THE analysis is as follows :—On Fig. 1, let P = the load on the ends of the column $A\,C\,B$, the central axis passing through the centre of gravity at any cross section. $S\,Q = y$, the deflection at S where $O\,Q = x, O\,A = O\,B = \frac{l}{2}$ it is assumed that y is small compared with l, the length of the column. Now $Py = M =$ the bending moment at S and $\frac{Py}{EI} = \frac{1}{R} = -\frac{d^2y}{dx^2}$ where E = Young's modulus of elasticity, R the radius of curvature of the central axis and of the neutral axis, and I the moment of inertia assumed constant,

then,
$$\frac{Py}{EI} = -\frac{d^2y}{dx^2}$$

or,
$$\frac{d^2y}{dx^2} + n^2y = o \ ; \quad \text{where} \quad n^2 = \frac{P}{EI}$$

the solution of this differential equation is

$$y = A \cos nx + B \sin nx$$

when $y = o$, $x = \frac{l}{2}$ or $-\frac{l}{2}$; this gives $B = 0$, therefore

$$y = A \cos nx = A \cos x \sqrt{\frac{P}{EI}}$$

when $x = \frac{l}{2}$, $y = o$. Hence

$$y = 0 = A \cos \frac{l}{2} \sqrt{\frac{P}{EI}}$$

giving $$\frac{l}{2}\sqrt{\frac{P}{EI}}=\frac{\pi}{2};\ \frac{3\pi}{2};\ \&c.,$$

using the minimum value $\frac{\pi}{2}$

$$P=\frac{\pi^2 EI}{l^2}$$

This is Euler's formula for columns. It requires correction in this respect. The value of R_0 is obtained on the assumption that the central and neutral axes of the column coincide. This is only true for pure bending. Actually the direct stress due to the load causes the neutral axis to shift to some distance from the central axis; the value of R—the radius of curvature of the neutral axis—is not then equal to R_0, the radius of curvature of the central axis. The correct solution, allowing for this variation of R and R_0, seems to have been first pointed out by the late John Neville, C.E., and is as follows:—On Fig. 2, consider the cross section $D\,E\,F\,G$ of a column (for convenience taken square) carrying a load P as in Fig. 1, the variation of stress on a cross-section would be similar to that shown by the figure $H\,I\,J\,K$ the position of the horizontal base $I\,J$, varying parallel to itself according as the bending and direct stresses varied on each cross-section. Let $M\,M$ be the central plane through the centre of gravity of the cross-section, and $N\,N$ the neutral plane and m the distances between them, then I° about the axis $M\,M=A\rho^2$ and I about $N\,N=A(\rho^2+m^2)$.

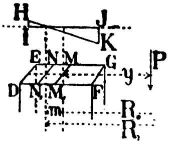

FIG. 2.

From Fig. (2) the following equations obtain

$$P=\frac{E}{R}mA \quad (1) \text{ Equation of forces}$$

and $$P\,(m+y)=\frac{EI}{R} \quad (2) \text{ Equation of moments.}$$

Solving between these two equations and substituting for I the following equation is obtained:—

$$my=\rho^2. \qquad (3)$$

As before, in Euler's analysis, $\frac{Py}{EI}=\frac{1}{R}$,

but $R=R_0+m$. Hence

$$\frac{Py}{EI}=\frac{1}{R_0+m},$$

Substituting from equation (3) for m the value $\frac{\rho^2}{y}$ and $P=pA$, the equation for R_0 becomes

$$\frac{1}{R_0}=\frac{1}{\rho^2}\,\frac{p}{E-p}\,y=-\frac{d^2y}{dx^2},$$

giving, as before,

$$\frac{d^2y}{dx^2}+a^2y=o, \text{ where } a^2=\frac{1}{\rho}\,\frac{p}{E-p}.$$

Proceeding to obtain the solution as previously we find

$$\frac{\pi}{2}=\frac{l}{2\rho}\sqrt{\frac{p}{E-p}},$$

or

$$p=\frac{E}{1+\frac{l^2}{\pi^2\rho^2}}, \text{ and } P=\frac{EA}{1+\frac{l^2}{\pi^2\rho^2}}.$$

It will be observed that in Euler's formula p becomes infinite as l vanishes, whereas in the corrected expression the limiting value of p is E. This formula, like Euler's, can only be considered as having an academic interest. It is important to note, however, that it is of the same form as the Rankine-Gordon formula.

This formula does not take into consideration the secondary flexure of the column, as will appear when this is taken into account. The limiting value of p is not E, but $\frac{Et}{\sqrt{3}\,r}$ where t is the thickness and r the mean radius of the column.

THE RANKINE-GORDON FORMULA.

The formula may be derived as follows :—Consider the column shown on Fig. 1, the bending moment at the middle is $M=P\delta=pA\delta$. From the theory of bending

$$\frac{M}{I}=\frac{p_{\prime}}{y}$$

Hence $pA\delta = \frac{p_{\prime} I}{y}$ or $\frac{p_{\prime}}{p} = \frac{\delta y}{\rho^2}$ A further deduction from the theory of bending shows that the proof deflection of a beam varies as $\frac{l^2}{y}$ or $\delta = \frac{l^2}{y}$ Where c is some constant substituting, the equation obtained is $\frac{p_{\prime}}{p} = c\frac{l^2}{\rho^2}$.

Add unity to both sides of this equation, then

$$\frac{p_{\prime} + p}{p} = 1 + c\frac{l^2}{\rho^2}.$$

Now, $p_{\prime} + p$ equals the sum of the stresses due to the load and to bending, and is equal to f the working strength of the material, also $p = \frac{P}{A}$, substituting these values and reducing, the formula takes the form

$$P = \frac{fA}{1 + c\frac{l^2}{\rho^2}}$$

For wrought iron columns with round ends $c = \frac{1}{9000}$ and for fixed ends $c = \frac{1}{36000}$

The manner of deriving the formula is open to criticism in this respect. The correct value of the coefficient c in the equation for the proof deflection is $\frac{1}{n}\frac{p_{\prime}}{E}$ where n is a numerical coefficient

Hence $$\frac{p_{\prime}}{p} = \frac{p_{\prime} l^2}{nE\rho^2} \text{ or } p = \frac{nE\rho^2}{l^2}.$$

That is Euler's formula. However, if the correction for the change in the deflection, due to the direct stress of the load, is made as in Neville's formula, the form of the Rankine-Gordon formula is correct, and gives close results compared with the values obtained experimentally. From the investigation of the foregoing formulæ it is evident that for long solid columns the load P approaches its Eulerian value, $\frac{\pi^2 EI}{l^2}$, and that for short columns the load P should approach the value F, the crushing strength of the material. To attain this result the foregoing formulæ can be put in the form

$$p=\frac{P}{A}=\frac{F}{1+\frac{F}{\pi^2 E}\left(\frac{l}{\rho}\right)^2},$$

which may be taken as true for both long and short columns. From a careful comparison of the published tests on solid column by Moncrieff and others the author is of the opinion that the value of c in the Rankine-Gordon formula for *solid* columns with round ends is closely given by the value of the coefficient $\frac{F}{\pi^2 E}$. It has already been pointed out that the failure of a column is complex, and that for the case of a solid column the breaking stress curve shows two well-defined humps, as shown on Plate I. On plotting values obtained from the Rankine-Gordon formula a second curve is obtained which cuts the experimental breaking stress curve once, and then approaches the Eulerian curve; and it will be noticed that the values obtained from this curve approximate closely to those given from the experimental curve.

The crushing strength of mild steel and wrought iron, as usually given, appear to be below the values obtained from the experiments on very short columns. It seems reasonable to expect that the crushing strength should be greater than the tensile strength, and the author has found, experimentally, that in some cases it is very much greater. When evaluating the constant $c=\frac{F}{\pi^2 E}$ it is necessary to bear this in mind. The effect of using a low value of F in practise causes the curves to start from a lower point, and the values obtained will be less than the experimental values—an error on the side of safety. The author, however, gives values later on of F, the strength to compression, deduced from the experiments published by Moncrieff and others.

MODIFIED RANKINE-GORDON FORMULA.

When a short tube is under the action of a load it has been shown experimentally that it breaks up into a number of waves. Consider now the half wave shown on Fig. 3. It may be regarded as being subjected to two different kinds of deformation, one due to the bending of the side of the tube as a thin column, and the other due to a direct thrust causing the stretching or compressing of the circumference of the tube into the wave form. Now, the applied load is partly supported by one kind of deformation and partly by the other. It is required to determine the proportion between the loads.

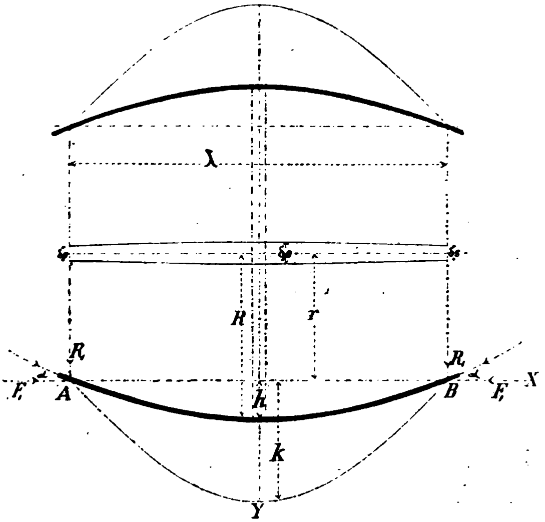

FIG. 3.

Consider a ring of the tube at the centre of the wave; owing to the stretching there is a circumferential tension in it of intensity : p; assume a normal load, q to be the cause of the stretching, then $qR = tp$ where R is the radius of the ring, and t its thickness. Assuming Hooke's law to hold true and that the wave is of a sinusoidal form, then, using the notation shown on the figure, the value at any section of $p = \frac{Eh}{r} \cos \frac{x}{k}$ where $E =$ Young's modulus of elasticity, and $R = r + h \cos \frac{x}{k}$ Hence at any ring

$$q = \frac{\frac{Eth}{r} \cos \frac{x}{k}}{r + h \cos \frac{x}{k}}$$

This gives the value of the normal load per unit of area on the tube which would cause deformation into the wave form. Now, consider a strip of breadth δs at r and $\delta_{,}s$ at any other section, then

$$\frac{\delta s}{\delta_{,}s} = \frac{r}{r + h \cos \frac{x}{k}}$$

it is required to determine the value of the total load on this strip $= \Sigma q \delta_{,} s \delta x$. From the above equations

$$q\delta_{,}s = \frac{\frac{Eth}{r} \cos \frac{x}{k}}{r + h \cos \frac{x}{k}} \; \frac{r + h \cos \frac{x}{k}}{r} \delta s$$

$$= \frac{Eth}{r^2} \cos \frac{x}{k} . \delta s$$

the summation of this along the length of the strip is

$$\Sigma(q\delta_{,}s)\ \delta x = \frac{Eth}{r^2} \delta s \int_{-\lambda/2}^{\lambda/2} \cos \frac{x}{k}\ \delta x$$

$$= \frac{Eth}{r^2} \delta s \frac{2\lambda}{\pi}$$

for the whole circumference of the tube, $\Sigma \delta s = 2\pi r$. Hence the total normal load on the interior $= \frac{4Eth\lambda}{r}$

Consider now the load on a strip of breadth δs at $A\ B$. The load diagram is the same as that of the equilibrium curve, and it is distributed between the two points of support, A and B, for the strip δs. Hence $\delta R_1 = \frac{Eth\lambda}{\pi r^2} \delta s$, and for the whole circumference $R_1 = \frac{2Eth\lambda}{r}$

Now $\frac{R_1}{F_1} = \tan\alpha$ also $y = h\cos\frac{x}{k}$ and $\tan\alpha = \frac{dy}{dx} = \pm\frac{h}{k}$ when $x = \pm\frac{\lambda}{2}$ Hence $\frac{h}{k} = \frac{2Eth\lambda}{F_1 r}$ and $F_1 = \frac{2Etk\lambda}{r} = \frac{2Et\lambda^2}{\pi r}$

The total vertical load on the wave is made up of two parts F_1 just determined, and F_2 the load which it would support as a thin column. This is equal to $\frac{\pi^2 EI}{\lambda^2}$ by Euler's formula. Hence the total load

$$F = F_1 + F_2 = \frac{2Et\lambda^2}{\pi r} + \frac{\pi^2 EI}{\lambda^2} = 2Et\left(\frac{\lambda^2}{\pi r} + \frac{\pi^3 rt^2}{12\lambda^2}\right)$$

where $I = \frac{2\pi rt^3}{12}$; on differentiation with regard to λ and equating for a minimum

$$\frac{\delta F}{\delta\lambda} = 2\,Et\left(\frac{2\,\lambda}{\pi r} - \frac{2\,\pi^3 rt^2}{12\,\lambda^3}\right) = 0$$

giving $\lambda = \frac{\pi}{\sqrt[4]{12}}\sqrt{rt}$;

and on substitution $F = \frac{2E\pi t^2}{\sqrt{3}} = \frac{EA}{\sqrt{3}}\frac{t}{r}$

where $A = 2\pi rt$

$$\text{and } f = \frac{F}{A} = \frac{E}{\sqrt{3}}\frac{t}{r}$$

These equations determine F and λ on the assumption of elastic deformation. The equation for the length of the wave shows that λ is proportional to the square root of the sectional area of the column.

The problem can also be solved from the energy point of view. Let U be the total energy. It is made up of two parts—U_1 the energy due to bending of the side of the tube as a column, and U_2 the energy due to the stretching of the tube into the wave in this case

$$U=U_1+U_2=\pi E h^2\left(\frac{\pi^3 I}{2\lambda^4}+\frac{t}{r}\right)\int_{-\lambda/2}^{\lambda/2}\cos^2\frac{x}{k}\,dx$$

And for the energy to be a minimum this gives as before

$$\lambda=\frac{\pi}{\sqrt[4]{12}}\sqrt{rt}$$

The results just obtained show that the value $f=B\left(\frac{\rho}{t}\right)^2$ given from the experiments* should be $f=\frac{E}{\sqrt{3}}\left(\frac{t}{r}\right)=\frac{Et}{\sqrt{6}\,\rho}$. It has long been known that the Eulerian formula $P=\frac{\pi^2 EI}{l^2}$ only applies to long columns, and was of no practical use for the design of short columns. The reason is now clear. For short columns the value of the load P approaches $F=\frac{2E\pi t^2}{\sqrt{3}}$, and for columns in which the length of the wave λ is small compared with the length of the column l, then $P=\frac{\pi^2 EI}{l^2}$. These results can be put in the following form :—

$$P=\frac{\dfrac{2E\pi t^2}{\sqrt{3}}}{1+\dfrac{l^2}{\pi^2 EI}\,\dfrac{2E\pi t^2}{\sqrt{3}}}$$

reducing this gives

$$P=\frac{EA}{\dfrac{4\sqrt{3}\pi\rho^2}{A}+\dfrac{l^2}{\pi^2\rho^2}}$$

or

$$p=\frac{P}{A}=\frac{E}{\sqrt{6}\,\dfrac{\rho}{t}+\dfrac{l^2}{\pi^2\rho}}$$

This equation may be looked upon as the theoretical form of Neville's equation applied to hollow columns when secondary flexure is taken into consideration.

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* Proc. I. Mech. E. 1905.

Differentiating the denominator with respect to P and ρ and equating to zero, the condition obtained for a maximum is

$$\rho^2 = \frac{\sqrt{Al}}{2\sqrt{\pi\sqrt[4]{3}}}$$

and on substitution

$$P = \frac{\sqrt{\pi E}A^{3/2}}{4\sqrt[4]{3}l} \text{ and } A = \left(\frac{4\sqrt[4]{3}Pl}{\sqrt{\pi E}}\right)^{2/3}$$

this equation determines the theoretical economic dimensions of the column. For the same reasons that apply to Euler's formula for long columns the formula $P = \frac{EA}{\sqrt{3}}\left(\frac{t}{r}\right) = \frac{EA}{\sqrt{6}}\frac{t}{\rho}$ only applies to columns which are thin compared with the diameter. The equation neglects the area required to transmit the load and only considers bending. To allow for this it can be put in the form

$$f = \frac{F}{1 + \frac{\sqrt{6nF}}{E}\frac{\rho}{t}}$$

where F is the strength to compression of the material, f the load per unit area on the column, and n a constant.

For sections which approximate to a solid bar the value $\frac{\sqrt{6nF}}{E}\left(\frac{\rho}{t} - \cdot 5\right)$ should be used for $\frac{\sqrt{6nF}}{E}\left(\frac{\rho}{t}\right)$, for the usual sections in practise the ·5 is negligible.

Now, the Rankine-Gordon formula is of the form

$$p = \frac{P}{A} = \frac{f}{1 + \frac{mf}{\pi^2 E}\left(\frac{l}{\rho}\right)^2}$$

where m = a constant, on substitution for f, the following modified form already given is obtained :—

$$p = \frac{P}{A} = \frac{F}{1 + \frac{F}{E}\left[\sqrt{6n}\left(\frac{\rho}{t}\right) + \frac{m}{\pi^2}\left(\frac{l}{\rho}\right)^2\right]}$$

$$= \frac{F}{1 + k\frac{\rho}{t} + c\left(\frac{l}{\rho}\right)^2}$$

and on differentiation the condition obtained for a maximum is

$$\left(\frac{\rho}{t}\right)\left(\frac{\rho}{l}\right)^2=\frac{c}{k}$$

From an examination of the experiments on solid and other columns of mild steel, wrought and cast iron by Hodgkinson, Christie, the Author, and others, the best value of m to adopt in the above formula is unity, giving $c=\frac{F}{\pi^2 E}$ for columns with round ends, and $\frac{F}{4\pi^2 E}$ for columns with fixed ends. If now values of F and E for mild steel, wrought and cast iron are used, as given in the following table, formulæ are obtained which give fair average values compared with the results obtained from the experimental tests, and which agree closely in some cases with the constants already in use. The values of $\frac{\sqrt{6nF}}{E}=k$ cannot for the present be accurately determined for other than circular columns for these sections, the value of $n=20$ gives good results, and, on substitution $k=\frac{50F}{E}$; in the following table, the values of k for mild steel, wrought and cast iron have been calculated from the values of F and E therein adopted.

Table of Constants used in the Modified Rankine-Gordon Formula.

Material	Strength to compression F	Modulus of elasticity E	Round ended column c	Fixed ended column c	k
Mild Steel,	80000	30000000	$\frac{1}{4000}$	$\frac{1}{16000}$	$\frac{1}{8}$
Wrought Iron,	55000	26000000	$\frac{1}{4500}$	$\frac{1}{18000}$	$\frac{1}{10}$
Cast Iron,	110000	14000000	$\frac{1}{1300}$	$\frac{1}{5200}$	$\frac{2}{5}$

Consider now the Rankine-Gordon formula for wrought iron columns with round ends

$$p=\frac{P}{A}=\frac{36000}{1+\frac{1}{9000}\left(\frac{l}{\rho}\right)^2}$$

This may be looked upon as a particular case of the general formula

$$p=\frac{P}{A}=\frac{F}{1+k\frac{\rho}{t}+c\left(\frac{l}{\rho}\right)^2}$$

which, on substitution of the constants for wrought iron becomes

$$p=\frac{P}{A}=\frac{55000}{1+\frac{1}{10}\frac{\rho}{t}+\frac{1}{4500}\left(\frac{l}{\rho}\right)^2}$$

Assuming a value of $\frac{\rho}{t}$ about 8, the formula on reduction becomes nearly the same as the Rankine-Gordon formula just given. The author is of the opinion, after a critical examination of Hodgkinson's experiments, from which the constants in the Rankine-Gordon formula were determined, and from the results of other experiments, that the modified form of the Rankine-Gordon formula with the constants adopted may be considered to give values which approximate fairly well with those given experimentally. Reference has already been made with regard to the value of F that should be used. For wrought iron and mild steel the values commonly given do not greatly exceed the strength to compression above the yield point, and are below the real crushing strength of the material. For this reason the values of F adopted are somewhat higher than usual, and the author thinks the values could be further increased with safety. For cast iron the value of F is sensibly correct, experiments being far more consistent than in the case of the other materials.

The values of k apply only to circular sections. Further experiments indicate that for other sections the wave lengths λ are in general longer than with the circular sections, and the value of k increased; also that symmetrical sections are stronger than unsymmetrical sections, angle irons and the like always giving a low breaking strength. The difference between the breaking strength of a symmetrical and an unsymmetrical section can be shown by testing a thin circular column with another like column which has been cut longitudinally.

ECONOMIC BEAM.

An interesting deduction from these experiments on the waves forms, produced by direct compression, is the application to the case of an economic circular beam. Consider a solid round beam resting on supports at its two ends and loaded in the centre; the load it will carry can be calculated from the formula given the theory of bending

$$M=\frac{fI}{y}$$

where M = the moment tending to break the bar,

f = the strength to compression per unit of area of a wave-length,

I = the moment of inertia of the cross-section of the beam about a diameter,

y = the half depth of the beam.

Now, consider the beam to remain with the area of the cross-section constant, but to gradually change its section to a hollow circular cross-section, then, under these conditions, the value of f in the equation is no longer the strength to compression of the material, but will be given by the limiting load on one wave-length, and the equation will be of the form

$$M=\frac{F}{1+k\frac{\rho}{t}}\ \frac{I}{y}$$

and on reduction

$$M=\frac{FA}{\frac{\sqrt{2}}{\rho}+\frac{4\pi k\rho}{A}}$$

differentiating and equating to zero the condition obtained for a maximum is

$$\rho^2=\frac{\sqrt{2}A}{4\pi k}$$

let $k=\frac{1}{8}$ the value obtained for mild steel tubes, then on substitution

$$M=\frac{FA^{3/2}}{2.8}$$

That is the analysis which shows that there is a certain value of the diameter compared with the thickness when the load supported will be a maximum, and for the case considered the value of $\frac{d}{t}$ about $\frac{1}{20}$.

Some interesting experiments by Hodgkinson, showing the effect of secondary flexure due to bending on circular and rectangular sections, are given in "Clark's Britannia and Conway Tubular Bridges."

Plate I.

THE ECONOMIC DESIGN OF COLUMNS.

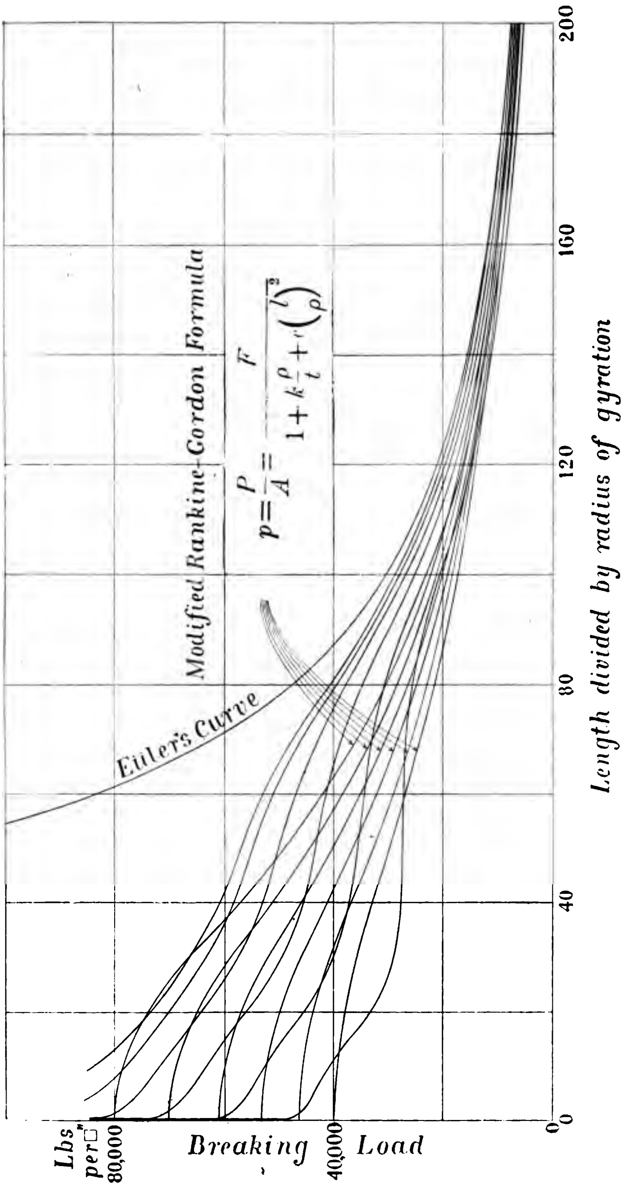

Diagram deduced from experiments on Mild Steel Tubes, showing the influence of thickness in determining their failure.

WEB STRESSES IN PLATE GIRDERS.

In the design of plate girders the practice is generally to consider the web of the girder as transmitting the shear stresses, and that the effect of the stiffeners is to stiffen the web to enable it to transmit these stresses. The disposition and the area of the cross-section of the stiffeners become under these conditions largely a matter of experience.

For girders of small span having few stiffeners, and in which the web is thick compared with the depth of the girder, the error involved is small, and does not cause much waste of material; when, however, larger spans come to be considered, the difficulty of determining the proportions of the web and stiffeners is accentuated. In such cases it is not sufficient to assume that the web wholly transmits the shear stress or that the function of the stiffeners is only to stiffen the web.

Some time ago the author enunciated the view that the stiffeners when placed on the web of a plate girder acted partly as struts, and that the web should be considered jointly with the struts if economy was to be attained. An alternative method to the usual procedure for the design of plate girders was then proposed, in which the web was assumed only to transmit tensile stresses, and the struts or stiffeners compressive stresses; it was then shown that the subject so considered led to a rational method of design of the girder, and did not leave the proportions for the web and struts so much to the individual judgment of the designer.

In order to test how far these views could be expected to hold true, the experiments about to be described were undertaken in the Engineering Laboratory of Trinity College, Dublin, the object of the research being to study the effect of various dispositions of stiffeners on the deformation of the web of the plate girder.

Since the classical experiments carried out on the model for the design of the Menai Tubular Bridge some sixty years ago, little research work in this direction appears to have been done, which seems unexplainable considering the importance of the subject and the enormous number of plate girders in existence. No doubt the proportions adopted in the majority of these girders are approximately correct; being the result of experience and the outcome of a long course of trial and error, it could hardly be otherwise. It must, however, be confessed that the proportions adopted, more especially of the web and stiffeners, are to a large extent empirical, and for this reason the subject requires further investigation.

A plate girder of the following dimensions was constructed:—Depth, 9½ inches; length, 5 feet 3 inches. The flanges were made up of two flat iron bars, 2 inches by ⅜ inch, and two angle-irons, 1¼ inches by ¼ inch. The framework of the girder was made in separate halves, and bolted together to the web. This construction allowed of different thicknesses of mild steel plate being used in the experiments. The arrangement of the girder and the method of testing are clearly shown in Fig. 1. One side of the angle-irons on the top and bottom flanges was cut at intervals along the length of the girder by vertical cuts. This was done to secure, as far as possible, flexibility of the flanges For testing the girder a vertical 10-ton Wicksteed testing machine was used, the concentrated load being applied centrally to the top flange of the girder, two vertical struts transmitting the load to the lower flange. In order to study the effect of placing stiffeners on the web, flat bars 1 inch by ½ inch were bolted together on each side of the web, the stiffeners being ¼ inch shorter than the depth between the edges of the angle-irons, thus ensuring that no stress could be directly transmitted by the angle-irons to the stiffeners.

A large number of tests were then carried out with different thicknesses of web and dispositions of stiffeners. The results of some of these tests were as follow:—

Plate Girder.—Length, 5 feet 3 inches; depth, 9½ inches.
Web, Mild Steel Plate, 28 S.W.G.

Figure number ...	1	2	3	4	5	6
Number of stiffeners	0	2	6	12	2	6
Total load in pounds	1500	2000	2500	3000	2500	3500

Web, Mild Steel Plate, 18 S.W.G.

Figure Number	7
Total load in pounds	12000

Plate Girder.—Length, 5 feet 3 inches; depth, 7½ inches.
Web, Mild Steel Plate, 24 S.W.G.

Figure number ...	8	9	10	11
Number of stiffeners	0	2	6	12
Total load in pounds	1800	2500	3300	4000

Web, Mild Steel Plate, 18 S.W.G.

Number of stiffeners	0	2	6	12
Total load in pounds	5500	8500	12000	16500

The exact determination of the loads causing distortion of the web was extremely difficult. As far as possible the numbers given refer to the loads just sufficient to produce the wave formation in the web, and from which an allowance requires to be made for the shear stress transmitted by the flanges. To obtain the blocks, Figs. 1 to 11, the loads were increased in order that the wave formation should be well defined; these show clearly the effect of the stiffeners in changing the shape and direction of the wave formation.

To enable a comparison to be made between the wave formation in the web of a built-up plate girder and of a rolled girder section, experiments were carried out on three different types of rolled section. The sections, after machining, were as follow:—

Figure	Flanges in. in.	Web in. in.
	$2 \times \frac{1}{4}$	$3\frac{1}{4} \times \frac{1}{16}$
12	$3 \times \frac{1}{8}$	$1\frac{1}{2} \times \frac{1}{16}$
13	$3 \times \frac{1}{4}$	$5 \times \frac{1}{16}$
14	$2 \times \frac{3}{16}$	$5 \times \frac{1}{16}$
15	$2 \times \frac{3}{16}$	$5 \times \frac{1}{16}$

In order to obtain the wave formation it was necessary in the larger sections, Figs. 13 and 15, to stiffen the web at the ends of the machined portion of the web with stiffeners—the distortion of the girder taking place sideways unless this was done. Fig. 15 shows this sideways distortion and its effect in changing the angle of inclination of the wave surface.

The following conclusions may roughly be drawn from these experiments:—The wave length of the wave formation set up in the buckling of the web is nearly independent of the thickness if the stiffeners are of great strength compared with the web. The angle of inclination of the waves depends upon the distance apart of the stiffeners and the depth of the girder. For a given value of $\frac{\text{distance apart of stiffeners}}{\text{depth of girder}}$ the angle of inclination is constant, and the smaller this ratio the smaller the angles of inclination with the vertical. The effect of the stiffeners is to change the direction of the stresses in the web, the part of the web between the stiffeners acting partly as a strut. The stiffeners prevent the formation of the waves, and severe local stresses are set up round the ends of the stiffeners, causing a crumpling up of this part of the web. Figs. 2 to 11 show the effect produced by this kind of distortion. The wave length depends on the angle of inclination of the wave formation in the web, and is roughly proportional to the length of the centre line of the wave measured between the stiffeners or flanges of the girder.

The permissible shear stress that should be adopted in the web of a plate girder, and the amount of its variation with the disposition of the stiffeners, have been much discussed. The ordinary method of procedure is usually to adopt some arbitrary value for the working shear strength of the material of the web; and the web, after design, is stiffened to enable it to transmit the shear stresses, some variation of the Rankine-Gordon formula being used for this purpose; this method gives roughly the disposition of the stiffeners, but in no way determines their cross section. Now, the experiments show that the stiffeners, when placed on the web, act partly as struts, and therefore require to be proportioned for the loads on them; in fact, the stiffeners play a double part—in stiffening the web against secondary flexure and in acting as struts.

For these reasons the author considers that the plate girder may best be looked upon as a dual sort of structure, in which the shear stresses are partly transmitted through the web, as in a rolled girder, and partly as in a triangulated girder. The change in the angle of the wave formation and the value of the total loads obtained as the distance apart of the stiffeners is varied is conclusive evidence that some such distribution of the stresses takes place.

The following solution of the permissible shear stress on the web of a plate girder the author believes to be new. Consider one of the waves shown on the girder in Fig. 1, and assume that it could be circumscribed by a rectangle, as shown in Fig. 16; also that it is under the action of a thrust of intensity p per unit of area along two parallel sides of the rectangle, and a pull of intensity q per unity of area along the other two sides of the rectangle. The wave surface may approximately be assumed to be sinusoidal, and its equation referred to the co-ordinate axes is

$$z=\delta \cos\frac{x}{k}\cos\frac{y}{h}$$

The plate is subjected to two different kinds of stresses, one of thrust and the other of pull, at right angles to each other. Consider now the plate as a series of parallel strips, such as *a, a;* these strips may be looked upon as forming a kind of spring arch, in which the thrust is partly resisted, as in a column, by bending, and partly resisted by direct thrust, as in an arch ring. Similarly, a series of strips, such as *b, b,* may be looked upon as a kind of spring suspension bridge, in which there is bending and tension—that is, the plate is in a state of strain of thrust and bending in one direction, and of bending and tension at right angles to it. It is required to determine the wave length λ in terms of L, the breadth for given values of p and q, and the thickness of the plate.

Assume that the strip *a, a* in its arched form is subjected to a load proportional to the area between the curve and its horizontal base. Take the origin of the co-ordinates at the centre of the base, and let

ω = density of unit volume of the load.
λ = span = wave length = $\pi\kappa$.
δ = rise of arch = deflection at the centre of the plate.

Then the intensity of the load at any point is

$$\omega\delta\cos\frac{x}{k}$$

and the shearing force at one support

$$F = \int_0^{\lambda/2} \omega\delta \cos \frac{x}{k} dx = \omega\delta\kappa \Big[\sin \frac{x}{k} \Big]_0^{\lambda/2} = \omega\delta\kappa$$

and the bending moment at the centre,

$$M = \int_0^{\lambda/2} \omega\delta\kappa \sin \frac{x}{k} dx = \omega\delta\kappa^2 \Big[\cos \frac{x}{k} \Big]_0^{\lambda/2} =$$

$$\omega\delta\kappa^2 = \frac{\omega\delta\lambda^2}{\pi^2}$$

and the horizontal thrust

$$P_2 = \frac{\omega\lambda^2}{\pi^2}$$

Now the strip *a, a* is also subjected to bending as a column; let P_1 be the load it would carry by Euler's formula $= \frac{\pi^2 EI}{\lambda^2}$ where E = Young's modulus of elasticity and I the moment of inertia. The total thrust $P = P_1 + P_2$; hence

$$\frac{1}{\lambda^2}\left(P - \frac{\pi^2 EI}{\lambda^2}\right) = \frac{\omega}{\pi^2} \quad . \quad . \quad (1)$$

Consider now the strip *b, b,* and imagine it loaded partly as a suspension bridge and partly as a beam, with a load of the same intensity ω as on the strip *a, a.* Let ω be that part of the load supported by bending, then the bending moment at the centre is $\frac{\omega_1 \delta L^2}{\pi^2}$. Now if the strip were loaded as a column, the bending moment at the centre would be $\frac{\pi^2 EI\delta}{L^2}$; these two expressions are assumed to be equal—that is,

$$\frac{\omega_1 L^2}{\pi^2} = \frac{\pi^2 EI}{L^2}$$

The other part of the load $\omega_2 = \omega - \omega_1$ is supported by the tension Q, and this is equal to $\frac{\omega_2 L^2}{\pi^2}$ giving

$$\frac{\omega}{\pi^2} = \frac{\omega_1 + \omega_2}{\pi^2} = \frac{1}{L^2}\left(Q + \frac{\pi^2 EI}{L^2}\right) \quad . \quad . \quad (2)$$

Substituting from equation (1) in equation (2) the equation obtained is

$$\frac{1}{\lambda^2}\left(P-\frac{\pi^2 EI}{\lambda^2}\right)=\frac{1}{L^2}\left(Q+\frac{\pi^2 EI}{L^2}\right) \quad . \quad . \quad (3)$$

The value of P per unit breadth equals pt, and of Q equals qt, where t equals the thickness of the plate, the moment of inertia $I=\frac{t^3}{12}$; also let $p=rq$, where r represents the ratio of the stresses to one another. Substituting these values in equation (3), the equation obtained is

$$p\left(\frac{1}{\lambda^2}-\frac{1}{rL^2}\right)=\frac{\pi^2 Et^2}{12}\left(\frac{1}{\lambda^4}+\frac{1}{L^4}\right) \quad . \quad . \quad (4)$$

Now let p and λ be the only variables in this equation, then, on differentiation and equating the result to zero, the condition obtained for a minimum is

$$\frac{\lambda^2}{L^2}=\frac{\sqrt{r^2+1}-1}{r} \quad . \quad . \quad . \quad (5)$$

That is, the wave length λ only depends on L and r. Substitute the value obtained for λ in equation (5) in equation (4), then

$$p=\frac{2\pi^2 Et^2}{12\lambda^2}=\frac{\pi^2 Et^2}{12L^2}\,\frac{2r}{\sqrt{r^2+1}-1} \quad . \quad . \quad (6)$$

showing that p varies as $\frac{t^2}{\lambda^2}$, also that the strength of the plate under compound stress of wave length λ is twice as strong as it would be when considered under direct compressive stress as a column. The expression obtained for p applies to the intensity of the load just sufficient to produce the wave surface; the value of the load to produce failure by crumpling of the plate is greater than this, owing to the ratio r of the stresses changing as the deformation goes on. This case differs from that of long columns in which a small increase of the load when near its maximum value produces failure.

For the same reasons that apply to Euler's formula for long columns equation (6) only applies to plates in which t is small compared with λ; in order to modify it for all cases it takes the form of the Rankine-Gordon formula for columns thus:—

$$p=\frac{F}{1+c\frac{\lambda^2}{t^2}} \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

where $c=\frac{12F}{2\pi^2E}$ and F is the strength per unit of area either to compression, pure shear, or tension, the values of which for mild steel or wrought iron are sensibly equal.

It is necessary here to remark that one of the arguments used in favour of the plate girder is that the material of the web is under tensile and compressive stresses at right angles to one another; and under such conditions it does double duty when compared with the struts and ties of the triangulated girder, and therefore should be more economical. The question then at once arises—In what way do these compound stresses affect the strength of materials such as wrought iron and mild steel? On this point little is really known. Experiments, so far, show that if the material is under a compressive stress, that a tensile stress at right angles to it does not sensibly affect it; and for materials in which the strength to compression is about equal to the strength to tension, that the strength to pure shear has nearly the same value.

The Rankine-Gordon formula as derived above has, with various values of c and F, long been used for stiffening the webs of plate girders. The formula being derived from experimental and theoretical work on columns, its use was naturally extended to the case of a buckled plate, and it is satisfactory to find that it has some theoretical basis.

The experiments are not of sufficient range to allow of any generalisation for determining λ in terms of L; also the problem is very complex, as the solution involves the slope of the waves, the ratio of the stresses, and the relative yielding of the web and stiffeners. For these reasons it is better to make some assumptions to determine the value of λ to be used in the formula, which will give a good approximation. On the usual hypothesis the waves are supposed to slope at 45°, and the stresses p and q to be equal to one another—that is, the web is assumed to be under a pure shear stress. The experiments certainly do not support this view, the stiffeners also act as struts, and therefore p must be less than q. For these reasons it is better to consider the girder as a dual sort of structure, in which the transmission of the stresses takes place partly as in a triangulated girder and partly as in a rolled girder.

Now, suppose the web to be designed first on the assumption that the web transmits tensile stresses only; and then, secondly, designed on the assumption that the web transmits the shear stress alone, and that the struts only act as stiffeners, then, if the thickness of the web and the distance apart of the stiffeners are the same for both designs, it would satisfy all the requirements from either point of view, and at the same time would lead to a rational method for the design of the struts.

Consider the girder shown on Fig. 17, let S be the shearing stress at the section A, A. Assume that the shear stress is wholly transmitted by the tensile stress in the web, and the lines of stress are at 45°; then, if $g =$ the factor of safety, $\frac{F}{g} = f$ the working strength to tension or compression of the material; also $f \sin 45° \cos 45° = f/_2$, and a the area of the cross section of the web $= Dt = \frac{2S}{f}$

It is now required to determine the distance apart of the stiffeners in order that the web may be sufficiently stiffened to enable it to wholly transmit the shear stresses as a pure shear stress; to do this requires that the value of λ in equation (7) should be determined.

The assumption of a pure shear stress would give $r = 1$, and on substitution in equation (5) $\lambda^2 = (\sqrt{2} - 1) L^2$. This view, however, cannot be maintained, as the experiments show that some of the shear stress is transmitted as in a triangulated girder, and the remainder as in the web of a rolled girder. It seems reasonable to suppose that the shear stresses should be nearly equally divided between the two structures if the proportions of the web and struts were not abnormal. Under these conditions the value of r, the ratio of the stresses, would be from one-third to one-half. Assuming one-half to be the best value to adopt, and substituting in equation (5), gives $\lambda = 0.5\ L$ nearly. Also the waves are assumed to slope at 45°, giving $L = \sqrt{2}d$, and on substitution $\lambda^2 = 0.5\ d^2$. Equation (7) then takes the form

$$p = \frac{F}{1 + \frac{3Fd^2}{\pi^2 E\ t^2}}$$

For ordinary solid columns of wrought iron $F = 60,000$ lbs. per square inch, and $E = 27,000,000$, giving

$$\frac{F}{\pi^2 E} = \frac{1}{4500} \text{ nearly,}$$

and on substitution

$$p = \frac{f}{1 + \frac{1}{1500}\frac{d^2}{t^2}} \quad . \quad . \quad . \quad (8)$$

where $f = \frac{F}{g}$, g being the factor of safety. Now assume that the stress in the web is a pure shear stress, $p = q = s$, and $stD = S$, giving on substitution in equation (8),

$$s = \frac{S}{Dt} = \frac{f}{1 + \frac{1}{1500}\frac{d^2}{t^2}} \quad . \quad . \quad . \quad (9)$$

The author is given to understand that the Rankine-Gordon formula in this form has already been applied to the design of plate girders. It is satisfactory to find that the empirical constant is more or less correct.

The web, as designed, has a thickness

$$t = \frac{2S}{fD} \text{ or } f/_2 = \frac{S}{t\,D}$$

Now if the web were subjected to a pure shear stress, $p = q = s = f/_2$, giving on substitution in equation (9)

$$\frac{1}{2}\left(1 + \frac{1}{1500}\,\frac{d^2}{t^2}\right) = 1,$$

or a ratio of $\frac{d}{t}$, about 40 for the distance apart of the struts compared with the thickness of the web, a result which compares favourably with existing practice. The dimensions of the struts now require to be determined. Assuming the web to transmit one-half of the shear stress by a pure shear stress, and the other half by a tensile stress, then the load on the strut $B\,B$ (Fig. 17) would be $p = \frac{S}{2}\,\frac{d}{D}$. Owing to the inclination of the waves varying as the distance apart of the struts is varied, the load on the strut for values of d greater than D is somewhat less than the above, and for values of d much smaller than D it is rather greater. For the usual values that obtain in practice the value obtained is sensibly correct.

Now the Rankine-Gordon formula for columns with round ends is

$$P = \frac{36{,}000A}{1 + \frac{1}{9000}\,\frac{l^2}{\rho^2}}$$

For struts, of irregular section, such as are used in girders, this formula* gives the best average results.

The value 36,000 is about $0.6F$ the strength to compression of wrought iron in pounds per square inch ; dividing by g gives $\frac{0.6F}{g} = 0.6f$; also $l = D$, and ρ is the radius of gyration of

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[*] Transactions of the Institution of Civil Engineers, Ireland, 1906.

the cross section of the strut, $P = \frac{S}{2}\frac{d}{D}$. On substitution of these values the formula becomes

$$\frac{S}{2}\frac{d}{D} = \frac{0.6fA}{1 + \frac{1}{9000}\frac{D^2}{\rho^2}}$$

If the value of $\frac{D}{\rho}$ is taken about 40, this gives $A = \frac{S}{f}\frac{d}{D}$ a cross section of the struts equal to one-half of the cross section of the web when $d = D$.

The web so designed satisfies the requirements when considered either from the view that the web takes the whole of the shear stresses as a pure shear stress, or that it transmits them only by a tensile stress; at the same time it leads to a rational method for the design of the struts. It will be found that the web, if judged by the assumption that the stresses are shared equally between the two parts of the dual structure, is about 25 per cent. stronger than necessary, and that a value of about $\frac{1.7S}{f}$ for the area of the cross section of the web would have been sufficient. Owing, however, to the uncertainty of the ratio r of the stresses, it seems better to proceed on the lines indicated, as the difference in the thickness of the web is only a small quantity.

There is yet to be considered an allowance for the stiffening of the girder to resist distortion sideways. The allowance for this depends upon the strength of the flanges to take up the small bending and twisting moments caused by the stresses not acting in the plane of the web, the flanges, together with the ends of the girder, tending to make an angle with one another, and show in plan roughly like a figure eight. To make allowance for this is difficult, and yet experiments show that it is very necessary. Suppose the flanges connected to the struts and web by ball-joints, then although the girder would be of sufficient strength to transmit the stresses in a vertical plane, it would be unstable; the ends of the girder, together with the gusset plates, prevent this relative movement by setting up torsion in the flanges. Usually the ends of the girder, together with gusset plates, at intervals, connect the flanges firmly together, their function being to play a double part in resisting the distortion sideways and the wind pressure. The flanges must also be of sufficient breadth and cross section to resist the twisting moments thus set up.

In girders of small span only few stiffeners are required, owing to the web being thick compared with the depth of the girder; also, for practical reasons, it is better to give a little extra material to the web and reduce the number of struts to a minimum. In large spans, however, owing to the web being relatively thin compared with the depth of the girders more stiffeners are necessary. The reason for this is that in railway and similar types of girders the total load varies approximately as the span. Now if the ratio of the depth to the span is assumed to be constant for different spans, then the thickness of the webs of these girders would be nearly the same throughout, and the number of stiffeners would be directly proportional to the span. The question then arises—What are the limiting spans for plate girders? When the value of $a = \frac{2S}{f}$ gives a thickness of the web less than from $\frac{3}{8}$ inch to $\frac{1}{2}$ inch it seems better to use triangulated members; for practical reasons it is undesirable to use plates of less thickness. If these conditions are granted, it will be found that about 60 feet to 80 feet is about the limit for the span of the plate girder, and for 80 feet to 100 feet spans compound girders are preferable, in which about one-third to one-half of the centre part of the span of the girder bracing is used, with web and struts for the remainder of the span; and for spans greater than these triangulated girders become necessary.

The process of evolution from the plate to the triangulated girder shows that for certain spans compound girders of the type mentioned are not only economical of material but have certain advantages for cheap construction, and it is surprising why they are not more used.

WEB STRESSES IN PLATE GIRDERS.

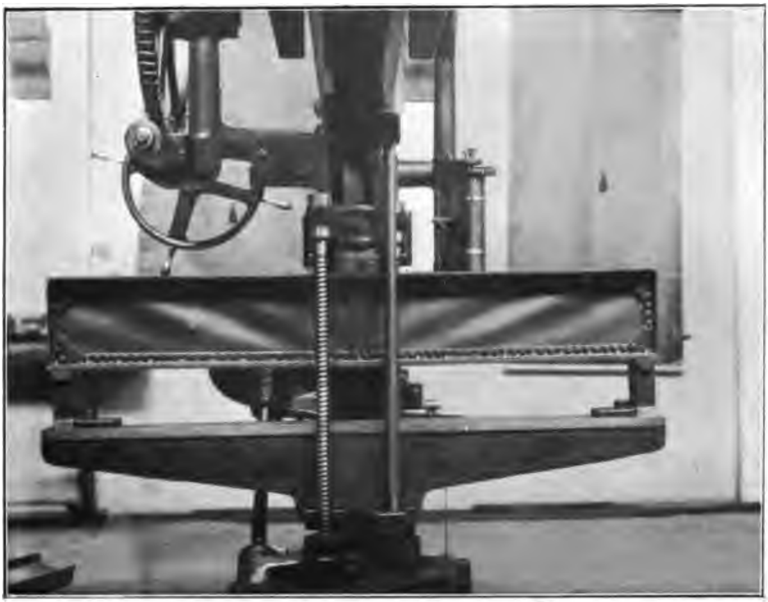

Fig. 1.

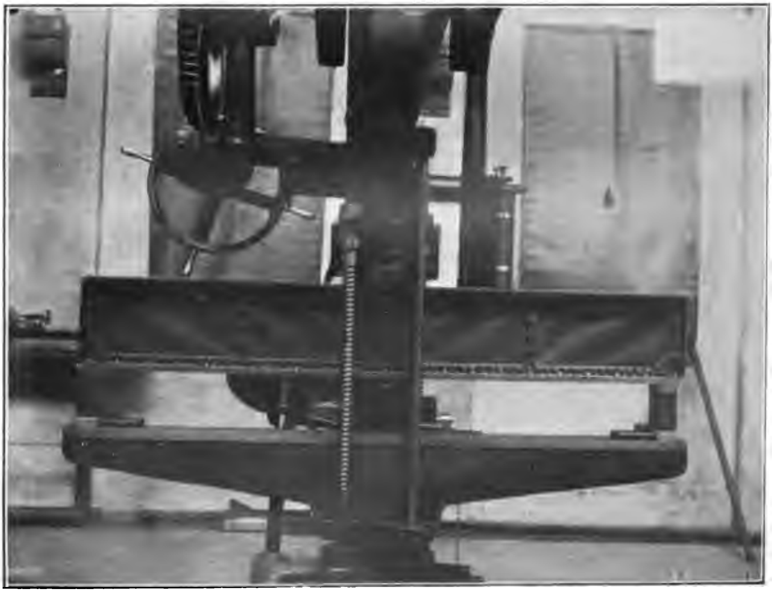

Fig. 2.

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ELEMENTARY
APPLIED MECHANICS.

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